Coupled dynamics of condensate and non-condensate in the trapped Bose gas

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Starting from first principles of the statistical mechanics we construct a dynamics of the condensed Bose gas at the nonzero temperatures. In particular, we derive an equation of motion for the condensate wavefunction and a quantum kinetic equation for the distribution function for the excited atoms. Obtained generalized Gross-Pitaevskii equation for condensate include effect of collisions with thermal cloud atoms (non-condensate). The Boltzmann quantum kinetic equation for non-condensate was obtained using Zubarev's method of nonequilibrium statistical operator. Therefore was obtained a complete dynamics of the condensed Bose gas at the nonzero temperatures.

1. Introduction

Bose-Einstein condensation (BEC) — one of the most interesting phenomena foreseen by quantum statistical mechanics. It is unique, because to be the exclusively quantum-statistical phase transmission and may occur without interaction between atoms. In the 1995 was got BEC "in a clean kind" in the trapped dilute ultracold clouds of alkaline metals [1–3].

Creation of the microscopic theory that described dynamics of condensed Bose gas is very important problem. The foundation of theoretical description such quantum degenerate systems with spontaneous broken symmetry were laid by Bogolyubov in the paper about weakly-nonideal Bose gas [4]. A central place in his theory occupies idea about separation of the condensate. To difference from Bogolyubov's model of weakly-nonideal Bose gas, the atomic Bose condensate is spatial inhomogeneous. At the temperatures close to zero $(0 - 0.4\,T_{BEC})$, when almost all atoms being in condensate, the dynamics of atomic Bose condensate is describing by nonlinear equation for condensate wave function — Gross-Pitaevskii (GP) equation. In this temperature region GP equation very precise describe the experimental date with condensate (condensate oscillation, interferences of condensates etc). At higher temperatures, when an appreciable fraction of atoms is excited out of the condensate, the dynamics of the trapped gas becomes much more complicated since it now involves the coupled motion of the condensate and the non-condensate degrees of freedom. Thus description of the atomic Bose condensate at the nonzero temperatures in terms of GP equation is inadequate.

In this paper, from the first principles, we obtain generalized GP equation, which include the non-condensate degrees of freedom. This equation isn't closed, because dynamics of the non-condensate atoms is indefinite. For description of the non-condensate atoms, using Zubarev's method of nonequilibrium statistical operator (NSO), we derived Boltzmann quantum kinetic equation for the distribution function for the excited atoms. Therefore we obtain complete dynamics of the condensed Bose gas at the nonzero temperatures.

2. Dynamics of the condensed Bose gas at nonzero temperatures

At very low temperatures, the dynamics of a trapped Bose gas is described by the time-dependent Gross-Pitaevskii (GP) equation [5] for the macroscopic wave function $\Phi(r, t)$ associated with the Bose condensate.

In terms of Bose quantum field operators, the many-body Hamiltonian describing interacting Bosons confined by an external potential $U_{\text{ext}}(\vec{r})$ is given by

$$H = \int d\vec{r}\,\psi^+(\vec{r})\left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\vec{r})\right)\psi(\vec{r})+$$

$$+\frac{1}{2}\int d\vec{r}d\vec{r}'\,V(\vec{r}-\vec{r}')\psi^+(\vec{r})\psi^+(\vec{r}')\psi(\vec{r}')\psi(\vec{r}). \tag{1}$$

In a dilute and cold gas, one can nevertheless obtain a proper expression for the interaction term by observing that, in this case, only binary collisions at low energy are relevant and these collisions are characterized by a single parameter, the s-wave scattering length, independently of the details of the two-body potential. This allows one to replace $V(\vec{r}-\vec{r}')$ with an effective interaction

$$V(\vec{r}-\vec{r}') = g\delta(\vec{r}-\vec{r}'),$$

where the coupling constant g is related to the scattering length a of the true potential through

$$g = \frac{4\pi\hbar^2 a}{m}.$$

The time-dependent Gross-Pitaevskii equation has next form:

$$i\hbar\frac{\partial\Phi(\vec{r},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\vec{r}) + gn_c\right)\Phi(\vec{r},t). \qquad (2)$$

Here $n_c(\vec{r},t) = |\Phi(\vec{r},t)|^2$ is density of atoms in the condensate.

The GP equation describes the motion of the condensate moving in the dynamic Hartree mean-field produced by the other atoms in the condensate and gives a closed equation for the order parameter $\Phi(\vec{r},t)$.

The GP equation (2) provides a very accurate description of the static and dynamic properties of a trapped Bose gas at low temperatures $T \leq 0.4 T_{BEC}$, as confirmed by many experiments in the last few years [5]. In superfluid ^{4}He, the non-condensate fraction at $T = 0$ is close to 90%. Thus in superfluid ^{4}He, one always has to deal with both the condensate and non-condensate atoms. Clearly a closed GP equation for $\Phi(\vec{r},t)$ is never valid in superfluid ^{4}He.

At higher temperatures (say $T > 0.4 T_{BEC}$), when an appreciable fraction of atoms is excited out of the condensate, the dynamics of the trapped gas becomes much more complicated since it now involves the coupled motion of the condensate and the non-condensate degrees of freedom.

To finding generalized GP equation we start with the usual Heisenberg equation of motion for the quantum field operator with the many-body Hamiltonian (1):

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(r)\right)\psi(\vec{r},t) +$$

$$+ g\psi^+(\vec{r},t)\psi(\vec{r},t)\psi(\vec{r},t).$$

Introducing the usual definition of the non-condensate field operator $\tilde{\psi}(\vec{r},t)$ according to

$$\psi(\vec{r},t) = \Phi(\vec{r},t) + \tilde{\psi}(\vec{r},t),$$

thus yields

$$i\hbar\frac{\partial\Phi(\vec{r},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\vec{r}) + gn_c(\vec{r},t) + 2g\tilde{n}(\vec{r},t)\right)\times$$

$$\times\Phi(\vec{r},t) + g\tilde{m}(\vec{r},t)\Phi^*(\vec{r},t) + g\langle\tilde{\psi}^+(\vec{r},t)\tilde{\psi}(\vec{r},t)\tilde{\psi}(\vec{r},t)\rangle,(3)$$

where $\tilde{n}(\vec{r},t) = \left\langle\tilde{\psi}^+(\vec{r},t)\tilde{\psi}(\vec{r},t)\right\rangle$ is the non-equilibrium non-condensate density, $\tilde{m}(\vec{r},t) = \left\langle\tilde{\psi}(\vec{r},t)\tilde{\psi}(\vec{r},t)\right\rangle$.

Equation (3) reduces to the GP equation (2) if all the atoms are in the condensate (i.e., $\tilde{n} = 0$) and the anomalous correlations ($\tilde{m}$ and $\langle\tilde{\psi}^+(\vec{r},t)\tilde{\psi}(\vec{r},t)\tilde{\psi}(\vec{r},t)\rangle$) are ignored. This is a very good approximation for $T \ll T_{\text{BEC}}$; at $T = 0$, the non-condensate fraction in trapped atomic gases is estimated to be less than 1% [6].

To derivation of quantum Boltzmann kinetic equation for the distribution function of the excited atoms consider equation of motion for $\tilde{\psi}(\vec{r},t)$

$$i\hbar\frac{\partial\tilde{\psi}}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\vec{r}) + 2gn\right)\tilde{\psi} - 2g\tilde{n}\tilde{\psi} + g\Phi^2\tilde{\psi}^+ +$$

$$+ g\Phi^*(\tilde{\psi}\tilde{\psi} - \tilde{m}) + 2g\Phi(\tilde{\psi}^+\tilde{\psi} - \tilde{n}) + g(\tilde{\psi}^+\tilde{\psi}\tilde{\psi} - \langle\tilde{\psi}^+\tilde{\psi}\tilde{\psi}\rangle),(4)$$

where $n = n_c + \tilde{n}$ is the total density. It will allow us to derive a kinetic equation for the excited atoms. Following Kirkpatrick and Dorfman [7], it is convenient to define the time evolution of $\tilde{\psi}$ by

$$\tilde{\psi}(\vec{r},t) = U^+(t,t_0)\tilde{\psi}(\vec{r},t_0)U(t,t_0),$$

where the unitary operator $U(t,t_0)$ evolves according to the equation of motion

$$i\hbar\frac{\partial U(t,t_0)}{\partial t} = H_{\text{eff}}(t)U(t,t_0).$$

Here, t_0 is the time at which the initial nonequilibrium density matrix $\hat{\rho}(t_0)$ is specified. The effective Hamiltonian in is given by

$$H_{\text{eff}}(t) = H_0(t) + H'(t), \qquad (5)$$

$$H'(t) = H_1'(t) + H_2'(t) + H_3'(t) + H_4'(t),$$

where the various contributions are defined as

$$H_0(t) = \int d\vec{r}\tilde{\psi}^+\left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\vec{r})\right)\tilde{\psi},$$

$$H_1'(t) = \int d\vec{r}\left(L_1(\vec{r},t)\tilde{\psi}^+ + L_1^*(r,t)\tilde{\psi}\right),$$

$$H_2'(t) = \frac{g}{2}\int d\vec{r}\left(\Phi^2(\vec{r},t)\tilde{\psi}^+\tilde{\psi}^+ + \Phi^{*2}(\vec{r},t)\tilde{\psi}\tilde{\psi}\right),$$

$$H_3'(t) = g\int d\vec{r}\left(\Phi^*(\vec{r},t)\tilde{\psi}^+\tilde{\psi}\tilde{\psi} + \Phi(\vec{r},t)\tilde{\psi}^+\tilde{\psi}^+\tilde{\psi}\right),$$

$$H_4'(t) = \frac{g}{2}\int d\vec{r}\tilde{\psi}^+\tilde{\psi}^+\tilde{\psi}\tilde{\psi} - 2g\int dr\tilde{n}(\vec{r},t)\tilde{\psi}^+\tilde{\psi},$$

$$U(\vec{r}) = U_{\text{ext}}(\vec{r}) + 2g\tilde{n}_c(\vec{r},t),$$

$$L_1(\vec{r},t) = -g\Big(\Phi^*(\vec{r},t)\tilde{m}(\vec{r},t) + 2\Phi(\vec{r},t)\tilde{n}(\vec{r},t) +$$

$$+ <\tilde{\psi}^+\tilde{\psi}\tilde{\psi}>\Big).$$

The expectation value of an arbitrary operator $O(t)$ which is made up of some combination of non-condensate field operators, with respect to the initial density matrix $\hat{\rho}(t_0)$ can be expressed as

$$\langle O(t) \rangle = \langle O \rangle^t = Sp\hat{\rho}(t_0)O(t) = Sp\tilde{\rho}(t,t_0)O(t_0),$$

where

$$\tilde{\rho}(t,t_0) = U(t,t_0)\hat{\rho}(t_0)U^+(t,t_0)$$

satisfies the following equation

$$\frac{\partial \tilde{\rho}(t,t_0)}{\partial t} + \frac{1}{i\hbar}[\tilde{\rho}(t,t_0), H_{\text{eff}}(t)] = 0.$$

Our ultimate objective is to obtain a quantum kinetic equation for the non-condensate atoms. For this purpose, we define the Wigner operator

$$\hat{f}(\vec{p},\vec{r},t_0) \equiv \int d\vec{r}' e^{\frac{i\vec{p}\vec{r}'}{\hbar}} \tilde{\psi}^+(\vec{r}+\frac{\vec{r}'}{2},t_0)\tilde{\psi}(\vec{r}-\frac{\vec{r}'}{2},t_0).$$

Its expectation value then yields the Wigner distribution function

$$f(\vec{p},\vec{r},t) = Sp\hat{f}(\vec{p},\vec{r},t_0)\tilde{\rho}(t,t_0).$$

Knowledge of this function allows one to calculate various nonequilibrium expectation values, such as the non-condensate density

$$\tilde{n}(\vec{r},t) = \int \frac{d\vec{p}}{(2\pi\hbar)^3} f(\vec{p},\vec{r},t).$$

The equation of motion for f is

$$\frac{\partial f(\vec{p},\vec{r},t)}{\partial t} = \frac{1}{i\hbar}Sp\tilde{\rho}(t)[f(\vec{p},\vec{r},t), H_{\text{eff}}(t)] =$$

$$= \frac{1}{i\hbar}Sp\tilde{\rho}(t)[f(\vec{p},\vec{r},t), H^0(t)] + \frac{1}{i\hbar}Sp\tilde{\rho}(t)[f(\vec{p},\vec{r},t), H'(t)]. \quad (6)$$

The first term on the right hand side of (6) defines the free-streaming operator in the kinetic equation. With the assumption that $U(\vec{r},t)$ varies slowly in space, we then have

$$\frac{\partial f(\vec{p},\vec{r},t)}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla f(\vec{p},\vec{r},t) - \nabla U \cdot \nabla_{\vec{p}}f(\vec{p},\vec{r},t) =$$

$$= \frac{1}{i\hbar}Sp\tilde{\rho}(t)[f(\vec{p},\vec{r},t), H'(t)].$$

The right hand side of this equation clearly represents the effect of collisions between the atoms.

The reduction of this term to the form of a binary collision integral performed by using method of NSO (see Appendix).

The collision integral is the sum of two contributions:

$$\left.\frac{\partial f}{\partial t}\right|_{\text{coll}} = C_{12}[f] + C_{22}[f],$$

where

$$C_{12}[f] = \frac{2g^2 n_c}{(2\pi)^2\hbar^4} \int d\vec{p}_2 \int d\vec{p}_3 \int d\vec{p}_4 \times$$

$$\times \delta(m\vec{v}_c + \vec{p}_1 - \vec{p}_2 - \vec{p}_3)\delta(\tilde{\varepsilon}_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}) \times$$

$$\times [\delta(\vec{p}-\vec{p}_1) - \delta(\vec{p}-\vec{p}_2) - \delta(\vec{p}-\vec{p}_3)] \times$$

$$\times [(1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3)] \quad (7)$$

describe collisions between thermal atoms, and

$$C_{22}[f] = \frac{2g^2}{(2\pi)^5\hbar^7} \int d\vec{p}_2 \int d\vec{p}_3 \int d\vec{p}_4 \times$$

$$\times \delta(\vec{p}+\vec{p}_2 - \vec{p}_3 - \vec{p}_4)\delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4}) \times$$

$$\times [(1+f)(1+f_2)f_3f_4 - ff_2(1+f_3)(1+f_4)] \quad (8)$$

describe collisions between thermal atoms and condensate atoms.

$$f = f(\vec{p},\vec{r},t), \quad f_i = f(\vec{p}_i,\vec{r},t).$$

Therefore quantum kinetic equation for the distribution function for the excited atoms

$$\frac{\partial f(\vec{p},\vec{r},t)}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla f(\vec{p},\vec{r},t) - \nabla U \cdot \nabla_{\vec{p}}f(\vec{p},\vec{r},t) =$$

$$= C_{22}[f] + C_{12}[f,\Phi]. \quad (9)$$

Generalized GP equation is as follows

$$i\hbar\frac{\partial \Phi(\vec{r},t)}{\partial t} = \Big(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\vec{r}) + gn_c(\vec{r},t) +$$

$$+2g\tilde{n}(\vec{r},t) - i\hbar R(\vec{r},t)\Big)\Phi(\vec{r},t), \quad (10)$$

where

$$R(\vec{r},t) = \int \frac{d\vec{p}}{(2\pi)^3} \frac{C_{12}[f,\Phi]}{2n_c(\vec{r},t)}.$$

The dissipative term R in (10) is associated with the exchange of atoms between the condensate and non-condensate, as described by the collision integral $C_{12}[f]$ in (7). We see that (9) and (10) must be solved self-consistently.

3. Conclusions

In this paper, we have derived a set of equations which describe the dynamics of a trapped Bose-condensed gas at finite temperatures. These equations consist of a generalized Gross-Pitaevskii equation for the condensate order parameter $\Phi(\vec{r}, t)$ and a semiclassical kinetic equation for the excited atom (non-condensate) distribution function $f(\vec{p}, \vec{r}, t)$. By limiting ourselves to higher temperatures, we arrive at a simple and intuitive picture in which the excited atoms behave as particles moving in a dynamic Hartree-Fock field. Collisions between all the atoms are included and in particular, allow for the transfer of atoms between the two components. Most of our discussion is devoted to the hydrodynamic regime and the derivation of a closed set of generalized hydrodynamic equations for the two components. These equations are based on the assumption that collisions between excited atoms are sufficiently rapid to drive the distribution function $f(\vec{p}, \vec{r}, t)$ toward the local equilibrium Bose-Einstein distribution $\tilde{f}(\vec{p}, \vec{r}, t)$, with the consequence that the $C_{22}[f]$ collision integral for the excited atoms vanishes. However, the $C_{12}[f]$ collision integral describing collisions between the condensate and noncondensate atoms remains finite, and as a result, our equations can be used to describe the situation in which the condensate is not in diffusive local equilibrium with the non-condensate.

Acknowledgements

Appendix

In the Appendix, we give detailed derivation of the expressions for the anomalous correlation functions $\langle \tilde{\psi}\tilde{\psi} \rangle$ and $\langle \tilde{\psi}^+\tilde{\psi}\tilde{\psi} \rangle$, as well as the collision integrals given by equations (7) and (8).

For this goal we use a Zubarev's method of nonequilibrium statistical operator (NSO)

The NSO method permits one to generalize the Gibbs ensemble method to the nonequilibrium case and to construct a nonequilibrium statistical operator which enables one to obtain the transport equations and calculate the kinetic coefficients in terms of correlation functions, and which, in the case of equilibrium, goes over to the Gibbs distribution.

In order to obtain a closed kinetic equation for the distribution function f we must express higher order correlation functions in terms of f. For this purpose, we treat $H'(t)$ as a perturbation to the zeroth-order Hamiltonian $H_0(t)$, as defined in (5). We shall effectively calculate collision integrals to second order in g.

Let us consider the Liouville equation with an infinitesimally small source into the right-hand side (we set $\hbar = 1$ throughout this Appendix)

$$\frac{\partial \rho(t)}{\partial t} + \frac{1}{i}[\rho(t), H(t)] = -\epsilon(\rho(t) - \rho_q(t)), \qquad (11)$$

where $\epsilon \to 0$ after the thermodynamic limit ($V \to \infty$, $N \to \infty$) and the quasi-equilibrium distribution is defined as

$$\rho_q = \exp\left\{ -\Phi(t) - \sum_n F_n(t)P_n \right\}.$$

Here $\Phi(t)$ – is a Massieu-Planck functional. The Lagrange multipliers F_m should be found from the self-consistency conditions

$$\langle P_m \rangle^t = \langle P_m \rangle_q^t,$$

where P_m are the set of a basic dynamic variables that describes of the nonequilibrium state.

In our case the Hamiltonian has the form (5)

$$H = H_0(t) + H'(t),$$

where $H'(t)$ we treat as a perturbation. In such situation useful a conversion to the interaction representation.

Let us now rewrite the Liouville equation (11) in the suitable to integration form

$$\left(\frac{\partial}{\partial t'} + \epsilon \right)\rho(t') + \frac{1}{i}[\rho(t'), H_0(t')] = \epsilon\rho_q(t') - \frac{1}{i}[\rho(t'), H'(t')].$$

Multiplying this equation by $\exp(\epsilon t')$ as well as by $U_0(t') = \exp\{-iH_0t'\}$ and $U_0^+(t') = \exp\{iH_0t'\}$ (from left and right respectively), integrating over time (from $-\infty$ to t), and using the property of the evolution operators, we find

$$\rho(t) = \epsilon \int_{-\infty}^{t} dt' \exp(-\epsilon(t - t'))U_0(t, t')\rho_q(t')U_0^+(t, t') -$$

$$- \int_{-\infty}^{t} dt' \exp(-\epsilon(t - t'))U_0(t, t')\frac{1}{i}[\rho(t'), H'(t')]U_0^+(t, t').$$

Here $U_0(t, t') = U_0(t)U_0^+(t')$.

After integrating of the term that proportional ϵ by parts obtain

$$\rho(t) = \rho_q(t) - \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} U_0(t,t') \left\{ \frac{\partial \rho_q(t')}{\partial t'} + \right.$$

$$+ \frac{1}{i}[\rho_q(t'), H_0(t')] \right\} U_0^+(t,t') - \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} \times$$

$$\times U_0(t,t') \frac{1}{i}[\rho(t'), H'] U_0^+(t,t'). \tag{12}$$

To find out how $\frac{\partial \rho_q(t)}{\partial t}$ is interaction depend. We have

$$\frac{\partial \rho_q(t)}{\partial t} = \sum_n \frac{\delta \rho_q(t)}{\delta \langle P_n \rangle^t} \frac{\partial \langle P_n \rangle^t}{\partial t} = \sum_n \frac{\delta \rho_q(t)}{\delta \langle P_n \rangle^t} \times$$

$$\times \left[i \sum_l \Omega_{nl}(t) \langle P_n \rangle^t + J_n(t) \right], \tag{13}$$

where Ω_{nl} satisfied the relationship

$$i \sum_l \Omega_{nl}(t) P_l = \frac{1}{i}[\rho_q(t), H_0(t)].$$

The $J_n(t)$ is defined as

$$J_n(t) = -i \langle [P_n, H'] \rangle^t = -i \mathrm{Sp}\Big([P_n, H'] \rho(t)\Big). \tag{14}$$

Taking into account relations (13) and (14) we obtain

$$\frac{\partial \rho_q(t)}{\partial t} - i[\rho_q(t), H_0(t)] = \sum_n \frac{\delta \rho_q(t)}{\delta < P_n >^t} J_n(t). \tag{15}$$

Substituting from (15) into (12) we find

$$\rho(t) = \rho_q(t) - \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} U_0(t,t') \times$$

$$\times \left\{ \sum_m \frac{\delta \rho_q(t')}{\delta < P_m >^{t'}} J_m(t') - i[\rho(t'), H'] \right\} U_0^+(t,t'). \tag{16}$$

Such form of the Liouville equation is useful to formulation of the perturbation theory.

Let us assume that

$$\rho(t) = \rho_q(t) + \sum_{k=1}^{\infty} \rho^{(k)}(t), \quad J_n(t) = \sum_{k=1}^{\infty} J_n^{(k)}(t).$$

Here $\rho^{(k)}$ and $J_n^{(k)}$ are proportional to g^k (g is a constant of interaction).

Notice that due the space homogeneity

$$J_n^{(1)}(t) = -i \langle [P_n, H'] \rangle_q^t = 0.$$

Iterating (16) to first order in $H'(t)$, one has

$$\rho(t) = \rho_q(t) - i \int\limits_{-\infty}^{t} dt' \exp(-\epsilon(t-t')) U_0(t,t') \times$$

$$\times [H'(t), \rho_q(t)] U_0^+(t,t'). \tag{17}$$

To a first approximation, for an arbitrary dynamical operator $O(t)$ the expectation value can be expressed as

$$\langle O(t) \rangle^t = \langle O(t) \rangle_q^t - i \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} \times$$

$$\times \left\langle [U_0(t,t') O(t) U_0^+(t,t'), H'(t)] \right\rangle_q^{t'}.$$

We first consider the three-field correlation function $\langle \tilde{\psi}^+ \tilde{\psi} \tilde{\psi} \rangle$. To lowest order in the interaction g, the three-field correlation function is given by

$$\left\langle \tilde{\psi}^+ \tilde{\psi} \tilde{\psi} \right\rangle^t = \left\langle \tilde{\psi}^+ \tilde{\psi} \tilde{\psi} \right\rangle_q^t - i \int_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} \times$$

$$\times \left\langle [U_0(t,t') \tilde{\psi}^+ \tilde{\psi} \tilde{\psi} U_0^+(t,t'), H'(t)] \right\rangle_q^{t'}. \tag{18}$$

The first term on the right hand side of (18) gives the initial correlation. Since the initial anomalous correlations regarding broken symmetry in the quasi-equilibrium statistical operator is absent, the term $\left\langle \tilde{\psi}^+ \tilde{\psi} \tilde{\psi} \right\rangle_q^t$ vanish. In addition, only those parts of H_1' and H_3' are kept which yield an equal number of creation and annihilation operators.

Thus

$$\left\langle \tilde{\psi}^+ \tilde{\psi} \tilde{\psi} \right\rangle^t = -i \int_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} \times$$

$$\times \left\langle [U_0(t,t') \tilde{\psi}^+ \tilde{\psi} \tilde{\psi} U_0^+(t,t'), H_1'(t) + H_3'(t)] \right\rangle_q^{t'}. \tag{19}$$

Suppose that the hydrodynamic variables vary slowly in space and time. Thus the terms contained in the effective Hamiltonian to come from values of $\vec{r}'$ and t' close to $\vec{r}$ and t.

Let us assume that

$$n_c(\vec{r}',t') \simeq n_c(\vec{r},t), \ \tilde{n}(\vec{r}',t') \simeq \tilde{n}(\vec{r},t), \ U(\vec{r}',t') \simeq U(\vec{r},t),$$

$$\theta(\vec{r}\,',t') \simeq \theta(\vec{r},t) + \frac{\partial\theta}{\partial t}(t'-t) + \nabla\theta\cdot(\vec{r}\,'-\vec{r}) =$$

$$= \theta(\vec{r},t) + \varepsilon_c(t'-t) + m\vec{v}_c\cdot(\vec{r}\,'-\vec{r}), \qquad (20)$$

$$U_0(t,t') = T_\tau \exp\left\{-i\int\limits_t^{t'} H_0(\tau)d\tau\right\} \simeq \exp\{-iH_0(t-t')\}.$$

Here $n_c(\vec{r},t) = |\Phi(\vec{r},t)|^2$ — density of condensate, θ is a phase of the macroscopic wavefunction $(\Phi(\vec{r},t) = |\Phi(\vec{r},t)|e^{i\theta(\vec{r},t)})$, $\vec{v}_c(\vec{r},t) = 1/m\nabla\theta(\vec{r},t)$ is a condensate velocity and $\varepsilon_c = \dfrac{\partial\theta}{\partial t} = \mu_c + \dfrac{mv_c^2}{2}$ is the local energy of a condensate atom.

Introducing the Fourier transform of the non-condensate field operators according to

$$\tilde{\psi}(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{p}} a_{\vec{p}}e^{i\vec{p}\vec{r}}, \quad \tilde{\psi}^+(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{p}} a_{\vec{p}}^+ e^{-i\vec{p}\vec{r}}$$

and using (20) and definition of the H_1', we obtain

$$\left\langle [U_0^+(t,t')\tilde{\psi}^+(\vec{r},t_0)\tilde{\psi}(\vec{r},t_0)\tilde{\psi}(\vec{r},t_0)U_0(t,t'), H_1'(t)]\right\rangle_q^{t'} \simeq$$

$$\simeq -2g\tilde{n}(\vec{r},t)n_c^{1/2}(\vec{r},t)\exp\{i\theta(\vec{r},t)\}\times$$

$$\times\frac{1}{V}\sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4} e^{-i(\vec{p}_{1c}+\vec{p}_1-\vec{p}_2-\vec{p}_3)\cdot\vec{r}}e^{i(\varepsilon_c+\tilde{\varepsilon}_1-\tilde{\varepsilon}_2-\tilde{\varepsilon}_3)(t-t')}\times$$

$$\times\delta_{\vec{p}_4,\vec{p}_c}\left(\langle a_{\vec{p}_1}^+ a_{\vec{p}_2}\rangle_q^{t'}\delta_{\vec{p}_3,\vec{p}_4} + \langle a_{\vec{p}_2}^+ a_{\vec{p}_3}\rangle_q^{t'}\delta_{\vec{p}_2,\vec{p}_4}\right), \qquad (21)$$

where $\vec{p}_c = m\vec{v}_c$, $\tilde{\varepsilon}_{p_i}(\vec{r}) = \dfrac{p_i^2}{2m} + U(\vec{r})$.

By analogy, using Wick's theorem for expressing of the higher order correlation functions we obtain

$$\left\langle [U_0^+(t,t')\tilde{\psi}^+(\vec{r},t_0)\tilde{\psi}(\vec{r},t_0)\tilde{\psi}(\vec{r},t_0)U_0(t,t'), H_3'(t)]\right\rangle_q^{t'} \simeq$$

$$\simeq 2g\tilde{n}(\vec{r},t)n_c^{1/2}(\vec{r},t)\exp\{i\theta(\vec{r},t)\}\times$$

$$\times\frac{1}{V^2}\sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3} e^{-i(\vec{p}_{1c}+\vec{p}_1-\vec{p}_2-\vec{p}_3)\cdot\vec{r}}e^{i(\varepsilon_c+\tilde{\varepsilon}_1-\tilde{\varepsilon}_2-\tilde{\varepsilon}_3)(t-t')}\times$$

$$\times\sum_{\vec{p}_1',\vec{p}_2',\vec{p}_3'} \delta_{\vec{p}_c+\vec{p}_1',\vec{p}_2'+\vec{p}_3'}\Big(\delta_{\vec{p}_2,\vec{p}_2'}\delta_{\vec{p}_3,\vec{p}_3'}\langle a_{\vec{p}_1}^+ a_{\vec{p}_1'}\rangle_q^{t'} + \delta_{\vec{p}_2,\vec{p}_2'}\langle a_{\vec{p}_1}^+ a_{\vec{p}_1'}\rangle_q^{t'}\times$$

$$\times\langle a_{\vec{p}_3}^+ a_{\vec{p}_3'}\rangle_q^{t'} + \delta_{\vec{p}_3,\vec{p}_3'}\langle a_{\vec{p}_1}^+ a_{\vec{p}_1'}\rangle_q^{t'}\langle a_{\vec{p}_2}^+ a_{\vec{p}_2'}\rangle_q^{t'} - \delta_{\vec{p}_1,\vec{p}_1'}\left\langle a_{\vec{p}_2}^+ a_{\vec{p}_2}\right\rangle_q^{t'}\times$$

$$\times\left\langle a_{\vec{p}_3}^+ a_{\vec{p}_3}\right\rangle_q^{t'} + \delta_{\vec{p}_2,\vec{p}_2'}\left\langle a_{\vec{p}_1}^+ a_{\vec{p}_3}\right\rangle_q^{t'}\left\langle a_{\vec{p}_3}^+ a_{\vec{p}_1'}\right\rangle_q^{t'} + \delta_{\vec{p}_3,\vec{p}_3'}\times$$

$$\times\left\langle a_{\vec{p}_1}^+ a_{\vec{p}_2}\right\rangle_q^{t'}\left\langle a_{\vec{p}_2}^+ a_{\vec{p}_2}\right\rangle_q^{t'}\right). \qquad (22)$$

Using (20), obtain

$$\left\langle a_{\vec{p}_i}^+ a_{\vec{p}_j}\right\rangle_q^{t'} = \mathrm{Sp}\Big(\rho_q(t_0)U_0^+(t',t_0)a_{\vec{p}_i}^+ a_{\vec{p}_j}U_0(t',t_0)\Big) \simeq$$

$$\simeq e^{i(\tilde{\varepsilon}_i - \tilde{\varepsilon}_j)}\left\langle a_{\vec{p}_i}^+ a_{\vec{p}_j}\right\rangle_q^{t_0} \qquad (23)$$

In view of a properties of the quasi-equilibrium statistical operator, we have

$$\left\langle a_{\vec{p}_i}^+ a_{\vec{p}_j}\right\rangle_q^{t_0} = \delta_{\vec{p}_i,\vec{p}_j}f(\vec{p}_i,\vec{r},t_0). \qquad (24)$$

Substituting from (21) and (22) into (19) and using (23) and (24), we obtain

$$\left\langle \tilde{\psi}^+(\vec{r},t)\tilde{\psi}(\vec{r},t)\tilde{\psi}(\vec{r},t)\right\rangle = -i2\pi g n_c^{1/2}e^{i\theta}\frac{1}{V^2}\times$$

$$\times\sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3}\left[\delta(\tilde{\varepsilon}_c + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3) + \frac{i}{\pi}P\frac{1}{\tilde{\varepsilon}_c + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3}\right]\times$$

$$\times\delta_{\vec{p}_c+\vec{p}_1-\vec{p}_2-\vec{p}_3}\Big[f_1(1+f_2)(1+f_3) - (1+f_1)f_3f_3\Big], \quad (25)$$

where $f_i = f(\vec{p}_i,\vec{r},t)$. We emphasize that in deriving (25), we used a Sokhotsky's formula

$$\int\limits_{-\infty}^{t} dt' e^{i(x-i\epsilon)(t-t')} = \pi\delta(x) + iP\left(\frac{1}{x}\right), \text{ if } \epsilon\to 0.$$

We note that the three-field correlation function in (25) is explicitly proportional to $\sqrt{n_c}$ and the interaction strength g. Thus it vanishes above T_{BEC} as well as in a non-interacting Bose gas.

Using similar techniques and approximations, we evaluate the remainder quantities

$$\tilde{m}(\vec{r},t) = \left\langle \tilde{\psi}(\vec{r},t)\tilde{\psi}(\vec{r},t)\right\rangle = -i\pi g\Phi\frac{1}{V}\sum_{\vec{p}_1,\vec{p}_2}\delta_{\vec{p}_1+\vec{p}_2,2\vec{p}_c}\times$$

$$\times[1 + f_1 + f_2]\left[\delta(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 - 2\tilde{\varepsilon}_c) + \frac{i}{\pi}P\frac{1}{\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 - 2\tilde{\varepsilon}_c}\right].$$

For collision integrals we find results (7) and (8).

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ОБ'ЄДНАНА ДИНАМІКА КОНДЕНСАТУ ТА НАДКОНДЕНСАТУ В ЗАХОПЛЕНОМУ БОЗЕ-ГАЗІ

П. П. Шигорін, А. В. Свідзинський

Р е з ю м е

Виходячи з перших принципів статистичної механіки ми побудували динаміку конденсованого бозе-газу при температурах відмінних від нуля. Зокрема, ми вивели рівняння руху для хвильової функції конденсату та квантове кінетичне рівняння для функції розподілу збуджених атомів. Одержане узагальнене рівняння Гросс-Пітаєвського для конденсату враховує зіткнення з надконденсатними атомами. Квантове кінетичне рівняння Больцмана для надконденсату отримано за допомогою метода Зубарєва нерівноважного статистичного оператора. Таким чином було отримано цілісну динаміку конденсованого бозе-газу при температурах відмінних від нуля.

ОБЪЕДИНЁННАЯ ДИНАМИКА КОНДЕНСАТА И НАДКОНДЕНСАТА В ПЛЕНЁННОМ БОЗЕ-ГАЗЕ

П. П. Шигорин, А. В. Свидзинский

Р е з ю м е

Исходя из первых принципов статистической механики мы сконструировали динамику конденсированного бозе-газа при отличных от нуля температурах. В частности, мы вывели уравнение движения для волновой функции конденсата, а также квантовое кинетическое уравнение для функции распределения возбужденных атомов. Полученное обобщенное уравнение Гросс-Питаевского для конденсата учитывает столкновения с надконденсатными атомами. Квантовое кинетическое уравнение Больцмана для надконденсата получено с помощью метода Зубарева неравновесного статистического оператора. Таким образом, было получено целостную динамику конденсированного бозе-газа при температурах, отличных от нуля.