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MATHEMATICS AND STATISTICS FOR AN INTERNATIONAL ECONOMIST:
some elements of Calculus: methodological instructions

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Annotation: The methodological recommendations are given curriculum for the discipline "Mathematics and statistics for an international economist" of the professional training of the bachelor of specialty 292 International Economic Relations.
The methodological recommendations contain some specific topics from Calculus of the course Mathematics and statistics for an international economist. For each of the topics typical examples along with their solutions are considered. They could be useful in preparation for practical classes as well as the independent work and also for quizzes and the final exam.

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Lesson topic: 1. Limit of a Function

1. Question Number: 1
Desc: Find the limits of a rational function by comparing the highest powers

Which of the following statements are true?
Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $\lim\limits_{x\to\infty}\dfrac{3x^2+x+2}{x^2+4x-1}=3$

2. $\lim\limits_{x\to\infty}\dfrac{2x^3+4x-12}{3x^2+x}=\infty$

3. $\lim\limits_{x\to\infty}\dfrac{1+5x-x^5}{2x^3}=0$

4. $\lim\limits_{x\to\infty}\dfrac{6+5x^2}{3x^2+7}=\dfrac{6}{7}$

5. $\lim\limits_{x\to\infty}\dfrac{x^6+3x^4+1}{x^7+7x^3+1}=0$

6. $\lim\limits_{x\to\infty}\dfrac{4x^2+x+12}{8x^3-3x+11}=\dfrac{1}{2}$

Solution:

$$\lim_{x\to\pm\infty}\left(\frac{a_n x^n+a_{n-1}x^{n-1}+\ldots+a_0}{b_m x^m+b_{m-1}x^{m-1}+\ldots+b_0}\right)=\begin{cases}\pm\infty & \text{if } n>m\\ \dfrac{a_n}{b_n} & \text{if } n=m\\ 0 & \text{if } n<m\end{cases}$$

Answer: 125

2. Question Number: 2
Desc: Use factorization to calculate the limit of an algebraic function

What whole number is equivalent to the value of $\lim\limits_{x\to 16}\dfrac{x^2-256}{\sqrt{x}-4}$?

Solution:

$$\lim_{x\to 16}\frac{x^2-256}{\sqrt{x}-4}=\lim_{x\to 16}\frac{(x-16)(x+16)}{\sqrt{x}-4}=\lim_{x\to 16}\frac{\cancel{(\sqrt{x}-4)}(\sqrt{x}+4)(x+16)}{\cancel{\sqrt{x}-4}}$$

$$=\lim_{x\to 16}(\sqrt{x}+4)(x+16)=(8)(32)=256$$

Answer: 256

3. Question Number: 3
Desc: Find the limit of the quotient of trigonometric functions

Which of the following limits has the same value as $\lim_{x\to 0}\frac{2-2\cos x}{\sin 2x}$?

Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $\lim_{x\to 0}\frac{\sin x}{\cos x(1+\cos x)}$

2. $\lim_{x\to 0}\frac{\cos x}{1+\cos x}$

3. $\lim_{x\to 0}\frac{2}{1+\cos x}$

4. 1

5. $\frac{1}{2}$

6. 0

Solution:

$$\lim_{x\to 0}\frac{2-2\cos x}{\sin 2x}=\lim_{x\to 0}\frac{\cancel{2}(1-\cos x)}{\cancel{2}\sin x\cos x}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{\sin x\cos x(1+\cos x)}=\lim_{x\to 0}\frac{1-\cos^2 x}{\sin x\cos x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\overset{\sin x}{\cancel{\sin^2 x}}}{\cancel{\sin x}\cos x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{\cos x(1+\cos x)}=\frac{0}{2}=0$$

According to the solution above we conclude that the correct statements are 1 and 6. Therefore, the answer is 16.

4. Question Number: 4
Desc: Apply the rule for the limit of sinx/x

What whole number is equivalent to the value of $\lim_{x\to 0}\frac{5\sin x+\sin 10x}{5x}$?

Solution:

$$\lim_{x\to 0}\frac{5\sin x+\sin 10x}{5x}=\lim_{x\to 0}\frac{5\sin x}{5x}+\lim_{x\to 0}\frac{\sin 10x}{5x}=\lim_{x\to 0}\frac{\sin x}{x}+\lim_{x\to 0}2\left(\frac{\sin 10x}{10x}\right)$$

$$=\lim_{x\to 0}\frac{\sin x}{x}+2\lim_{10x\to 0}\left(\frac{\sin 10x}{10x}\right)=1+2(1)=3$$

Answer: 3

5. Question Number: 5
Desc: Apply the rule for the limit of tanx/x

If $\lim_{x\to 0}\frac{\tan 18x}{8x}=\frac{k}{4}$, where k is a whole number, what is the value of k?

Solution:

$$\lim_{x\to 0}\frac{\tan 18x}{8x}=\lim_{x\to 0}\frac{\frac{\sin 18x}{\cos 18x}}{8x}+\lim_{x\to 0}\left(\frac{1}{\cos 18x}\right)\left(\frac{\sin 18x}{8x}\right)=\lim_{x\to 0}\left(\frac{1}{\cos 18x}\right)\left(\frac{18}{8}\right)\left(\frac{\sin 18x}{18x}\right)$$

$$=\left(\frac{1}{1}\right)\left(\frac{18}{8}\right)(1)=\frac{18}{8}=\frac{9}{4}$$

Answer: 9

6. Question Number: 6
Desc: Apply the epsilon-delta continuity condition

Function f defined by $f(x) = 5x - 6$ is continuous at $x = 0$ because for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x)+6|<\varepsilon$ whenever $|x|<\delta$. The largest value of δ when $\varepsilon = 0.1$ is $\frac{1}{k}$, where k is a whole number. What is the value of k?

Solution:
The value of δ must insure that $|5x - 6 + 6| < \varepsilon$.
Simplifying $|5x - 6 + 6| < \varepsilon$ gives $|x| < \frac{\varepsilon}{5}$.
Taking δ to be $\frac{\varepsilon}{5}$, substituting 0.1 for ε and simplifying yields, $\delta = \frac{1}{50}$.
Therefore, the value of *k* is 50.

7. Question Number: 7
Desc: Apply the condition of the existence of the limit to find a constant

Consider the function $f(x)=\begin{cases} k+x & \text{if} \quad x<-3 \\ 4x^2 & \text{if} \quad x\ge -3 \end{cases}$.

For what value of k does $\lim\limits_{x\to -3} f(x)$ exist?

Enter your answer as a whole number.

Solution:

$\lim\limits_{x\to -3^-} f(x) = \lim\limits_{x\to -3^-} (k + x) = k - 3$ **and** $\lim\limits_{x\to -3^+} f(x) = \lim\limits_{x\to -3^+} 4x^2 = 36$.

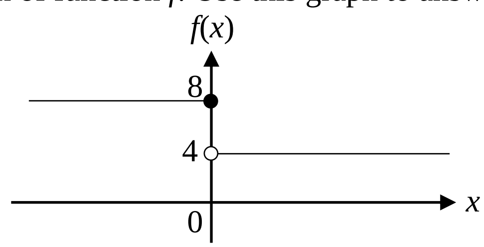
$\lim\limits_{x\to -3} f(x)$ **exists if and only if** $\lim\limits_{x\to -3^-} f(x) = \lim\limits_{x\to -3^+} f(x)$ **which implies** $k - 3 = 36$ **or** $k = 39$.

Therefore, the value of *k* is 39.

8. Question Number: 8

Desc: Identify the limits of a piecewise function defined graphically

Below is the graph of function *f*. Use this graph to answer the following question.



Which of the following statements are true?

Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $\lim\limits_{x\to 0^-} f(x) = 8$

2. $\lim\limits_{x\to 0} f(x) = 6$

3. $\lim\limits_{x\to 0^+} f(x) = 4$

4. $\lim\limits_{x\to 0} f(x)$ does not exist.

5. $\lim\limits_{x\to 0^-} f(x)$ does not exist.

6. $\lim\limits_{x\to 0^+} f(x)$ does not exist.

Solution:

$\lim_{x\to 0^-} f(x) = 8$ **and** $\lim_{x\to 0^+} f(x) = 4$.

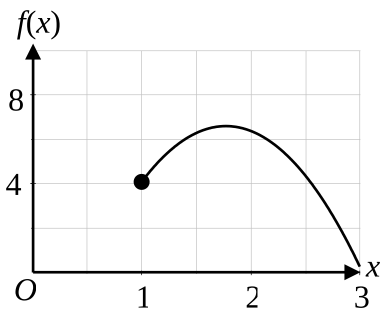
The one-sided limits exist.

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ **we conclude that** $\lim_{x\to 0} f(x)$ **does not exist.**

Therefore, the statements 1, 3 and 4 are correct and the answer is 134.

9. Question Number: 9

Desc: Apply the condition of the existence of the limit of a function from a graph



Function f is defined for $x \geq 1$ by the graph above. Function g is defined for $x < 1$ such that $\lim_{x\to 1^-} g(x)$ exists. Function h is defined over the set of real numbers by

$$h(x) = \begin{cases} g(x) & \text{when } x < 1 \\ f(x) & \text{when } x \geq 1 \end{cases}.$$

If $\lim_{x\to 1} h(x)$ exists, what must be the value of $\lim_{x\to 1^-} g(x)$?

Enter your answer as a whole number.

Solution:

$\lim_{x\to 1} h(x)$ **exists implies** $\lim_{x\to 1^-} h(x) = \lim_{x\to 1^+} h(x)$ **or** $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} f(x)$.

From the graph above we conclude that $\lim_{x\to 1^+} f(x) = 4$. **Therefore,** $\lim_{x\to 1^-} g(x) = 4$.

Lesson topic: 2. Continuity

1. Question Number: 10
Desc: Check the continuity of a piecewise function that has a jump

Consider the function $f(x)=\begin{cases} x+1 & \text{if } x\le -2 \\ x+2 & \text{if } -2<x<2 \\ 2x & \text{if } x\ge 2 \end{cases}$.

Which of the following statements are true?
Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $\lim\limits_{x\to -2^+} f(x) = f(-2)$

2. $\lim\limits_{x\to -2^+} f(x) = f(-2)$

3. $f(x)$ **is right continuous at** $x = -2$.
4. $f(x)$ **is left continuous at** $x = -2$.
5. $f(x)$ **is continuous at** $x = 2$.
6. $f(x)$ **is discontinuous at** $x = 2$.

Solution:

$\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-}(x+1) = -1 = f(-2)$. **So,** $f(x)$ **is left continuous at** $x = -2$.

$\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+}(x+2) = 0 \ne -1 = f(-2)$. **So,** $f(x)$ **is not right continuous at** $x = -2$.

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(x+2) = 4$ **and** $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} 2x = 4$.

Since $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2) = 4$ **then** $f(x)$ **is continuous at** $x = 2$.
Therefore, the statements 1, 4 and 5 are correct and the answer is 145.

2. Question Number: 11
Desc: Verify the continuity of a piecewise function

Consider the function $f(x) = |x - 3|$.

Which of the following statements are true?
Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $f(3)$ exists.

2. $\lim\limits_{x\to 3^+} f(x)$ exists.

3. $\lim\limits_{x\to 3^-} f(x)$ does not exist.

4. $\lim\limits_{x\to 3} f(x)$ exists.

5. $f(x)$ is continuous at $x = 3$.

6. $f(x)$ has a removable discontinuity at $x = 3$.

Solution:
Express the function *f* in the following form

$$f(x)=\begin{cases} x-3 & \text{when } x\ge 3 \\ -x+3 & \text{when } x<3 \end{cases}.$$

$f(3) = 3 - 3 = 0$ implies $f(3)$ exists.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x-3) = 0 \text{ implies } \lim_{x\to 3^-} f(x) \text{ exists.}$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (-x+3) = 0 \text{ implies } \lim_{x\to 3^+} f(x) \text{ exists.}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \text{ implies } \lim_{x\to 3} f(x) \text{ exists.}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \text{ implies } f(x) \text{ is continuous at } x = 3.$$

Therefore, the only statements 1, 2, 4 and 5 are correct and the answer is 1245.

3. Question Number: 12
Desc: Verify the continuity of a piecewise function

Consider the function $f(x)=\begin{cases} 3x & \text{if } x<-2 \\ 2x-2 & \text{if } x\ge -2 \end{cases}$.

Which of the following statements are true?
Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $f(-2)$ exists.

2. $f(x)$ is not continuous at $x = -2$ because it has a gap.

3. $f(x)$ is continuous at $x = -2$ because $f(-2)$ exists.

4. $f(x)$ is not continuous at $x = -2$ because it has a jump.

5. $\lim_{x\to-2} f(x)$ exists.

6. $f(x)$ is continuous at $x = -2$ because $f(-2) = \lim_{x\to-2} f(x)$.

Solution:

$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} 3x = -6$ **and** $\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} (2x-2) = -6$

$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^+} f(x)$ **implies** $\lim_{x\to-2} f(x)$ **exists.**

$f(-2) = 2(-2) - 2 = -6$ implies $f(-2)$ exists.

$\lim_{x\to-2} f(x) = f(-2)$ **implies $f(x)$ is continuous at $x = -2$.**

Therefore, the only statements 1, 5 and 6 are correct and answer is 156.

4. Question Number: 13
Desc: Define a rational function at a point to satisfy the continuity condition

Consider the function $f(x) = \begin{cases} \dfrac{x^2-36}{x-6} & \text{when } x \neq 6 \\ m & \text{when } x = 6 \end{cases}$.

For what value of *m* is function *f* continuous over the set of real numbers? Enter your answer as a whole number.

Solution:

Since $f(x)$ is the quotient of two continuous functions then it is continuous over the set of real numbers except $x = 6$.

Consider the case $x = 6$.

$$\lim_{x\to6} f(x) = \lim_{x\to6} \frac{x^2-36}{x-6} = \lim_{x\to6} \frac{\cancel{(x-6)}(x+6)}{\cancel{x-6}} = \lim_{x\to6} (x+6) = 12$$

The function *f* is continuous at $x = 6$ if and only if $\lim_{x\to6} f(x) = f(6) = m$ **or $m = 12$.**

Therefore, the value of *m*, for which the function *f* is continuous over the set of real numbers, is 12.

5. Question Number: 14
Desc: Define a trigonometric function at a point to satisfy the continuity condition

Consider the function $g(x) = \begin{cases} \dfrac{\sin 12x}{3x} & \text{when } x \neq 0 \\ b & \text{when } x = 0 \end{cases}$.

What is the value of *b* that makes *g*(*x*) continuous over the set of real numbers? Enter your answer as a whole number.

Solution:
Since *g*(*x*) is the quotient of two continuous functions then it is continuous over the set of real numbers except *x* = 0.
Consider the case *x* = 0.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin 12x}{3x} = \lim_{x\to 0}\frac{4\sin 12x}{12x} = 4\lim_{12x\to 0}\frac{\sin 12x}{12x} = 4(1) = 4$$

The function *g* is continuous at *x* = 0 if and only if $\lim_{x\to 0} g(x) = g(0) = b$ or *b* = 4.
Therefore, the value of *b*, for which the function *g* is continuous over the set of real numbers, is 4.

6. Question Number: 15
Desc: Identify the points of discontinuities of a piece-wise function defined algebraically

How many points of discontinuity does the function $f(x) = \begin{cases} \dfrac{3x}{2x+4} & \text{when } x \le 1 \\ \dfrac{-6x}{x-3} & \text{when } x > 1 \end{cases}$ have?

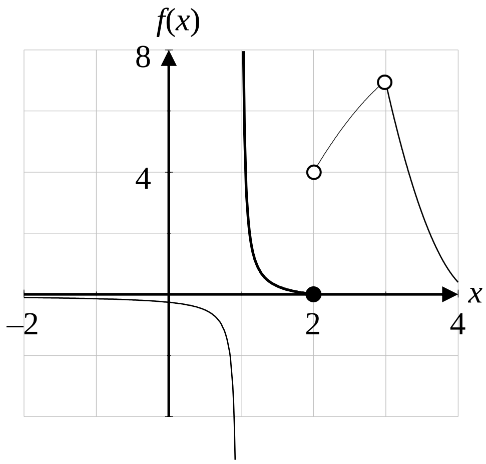
Solution:
The function *f* is not defined when 2*x* + 4 = 0 and *x* – 3 = 0 or *x* = –2 and *x* = 3. So, *x* = –2 and *x* = 3 are points of discontinuity.
Consider the case *x* = 1.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\frac{3x}{2x+4} = \frac{1}{2} \text{ and } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\frac{-6x}{x-3} = 3$$

Since $\lim_{x\to 1^-} f(x) \ne \lim_{x\to 1^+} f(x)$ then $\lim_{x\to 1} f(x)$ does not exist and function *f* is discontinuous at *x* = 1.
Therefore, the function *f* has 3 points of discontinuity.

7. Question Number: 16
Desc: Identify the points of discontinuity of a piece-wise function defined graphically

The figure below shows the graph of *f*(*x*). Use this graph to answer the following question.



How many points of discontinuity does the function *f* have?

Solution:

Consider $x = 1$: $\lim_{x\to 1^-} f(x) = -\infty \neq +\infty = \lim_{x\to 1^+} f(x)$. So, $x = 1$ is a point of discontinuity.

Consider $x = 2$: $\lim_{x\to 2^-} f(x) = 0 \neq 4 = \lim_{x\to 2^+} f(x)$. So, $x = 2$ is a point of discontinuity.

Consider $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ but $f(3)$ does not exist. So, $x = 3$ is a point of discontinuity.

Therefore, the function *f* has 3 points of discontinuity.

Lesson topic: 3. Definition of the derivative

1. Question Number: 17
Desc: Apply the continuity and differentiability conditions of a piecewise function

The function $f(x)=\begin{cases} ax^3+bx+11 & x<-1 \\ -2x+1 & x\ge -1 \end{cases}$ is differentiable at $x = -1$, what is the value of $f(-2)$?

Enter your answer as a whole number.

Solution:

$f(x)$ is differentiable implies $f(x)$ is continuous.

Since $f(x)$ is continuous then $\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^+} f(x)=f(-1)$ that yields,

$a(-1)^3+b(-1)+11=-2(-1)+1$ or $a + b = 8$.

Since $f(x)$ is differentiable then $\lim\limits_{h\to 0^-}\dfrac{f(h-1)-f(-1)}{h}=\lim\limits_{h\to 0^+}\dfrac{f(h-1)-f(-1)}{h}$.

$$\lim_{h\to 0^-}\frac{f(h-1)-f(-1)}{h}=\lim_{h\to 0^-}\frac{a(h-1)^3+b(h-1)+11-3}{h}$$

$$\lim_{h\to 0^-}\frac{ah\left(h^2-3h+3\right)+bh+(-a-b+8)^{=0}}{h}=\lim_{h\to 0^-}\left[a\left(h^2-3h+3\right)+b\right]=3a+b$$

$$\lim_{h\to 0^+}\frac{f(h-1)-f(-1)}{h}=\lim_{h\to 0^+}\frac{-2(h-1)+1-3}{h}=\lim_{h\to 0^+}\frac{-2h}{h}=-2$$

So, $3a + b = -2$.

Solve the system $\begin{cases} a+b=8 \\ 3a+b=-2 \end{cases}$.

Subtracting the first equation from the second one yields, $2a = -10$ or $a = -5$.
Substituting $a = -5$ into the first equation gives, $-5 + b = 8$ or $b = 13$.
So, the function $f(x)$ is

$$f(x)=\begin{cases} -5x^3+13x+11 & x<-1 \\ -2x+1 & x\ge -1 \end{cases}.$$

$f(-2) = -5(-2)^3+13(-2)+11=40-26+11=25$.

Therefore, the value of $f(-2)$ is 25.

2. Question Number: 18
Desc: Apply the product rule to find the derivative

Let $y = -3x\cot x$. Find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$, and enter the answer rounded to the nearest whole number.

Solution:

$$\frac{dy}{dx} = \left(-3x\cot x\right)' = \left(-3x\right)'\cot x - 3x\left(\cot x\right)' = -3\cot x + 3x\csc^2 x$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -3\cot\frac{\pi}{4} + 3\frac{\pi}{4}\csc^2\frac{\pi}{4} = -3(1) + \frac{3\pi}{4}\left(\sqrt{2}\right)^2 = -3 + \frac{3\pi}{2} \approx 1.71$$

Therefore, the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$, rounded to the nearest whole number, is 2.

3. Question Number: 19
Desc: Verify the continuity of a piecewise function

Consider the function $f(x) = \begin{cases} 4x+1 & \text{if } x \le 0 \\ 3x^2 & \text{if } x > 0 \end{cases}$.

Which of the following statements are true?
Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $f(x)$ is not differentiable at $x = 0$ because $f(0)$ does not exist.

2. $f(x)$ is not continuous but is differentiable at $x = 0$.

3. $f(x)$ is not differentiable at $x = 0$ because it is not continuous at that point.

4. $f(x)$ is differentiable at $x = 0$ because $f(0)$ exists.

5. $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h}$ does not exist.

6. $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h}$ exists.

Solution:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(4x+1) = 1 \text{ and } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x^2 = 0$$

$\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$ **implies *f*(*x*) is not continuous at *x* = 0. So, *f*(*x*) is not differentiable at *x* = 0 because it is not continuous at that point.**

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(4h+1)-1}{h} = \lim_{h\to 0^-} \frac{4\cancel{h}}{\cancel{h}} = 4$$

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{3h^2-1}{h} = \lim_{h\to 0^+} \left(3h-\frac{1}{h}\right) = -\infty$$

$\lim\limits_{h\to 0^-} \dfrac{f(h)-f(0)}{h} \neq \lim\limits_{h\to 0^+} \dfrac{f(h)-f(0)}{h}$ **implies** $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$ **does not exist.**

Therefore, the only correct statements are 3 and 5 and the answer is 35.

4. Question Number: 20
Desc: Find the derivative of a trigonometric function

If $y = 3\cot(\pi - 6x)$, what is the value of $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{24}$.
Enter the answer as a whole number.

Solution:

$$\frac{dy}{dx} = \left(3\cot(\pi-6x)\right)' = 3\left(-\csc^2(\pi-6x)\right)(\pi-6x)' = 18\csc^2(\pi-6x)$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{24}} = 18\csc^2\left(\pi-6\times\frac{\pi}{24}\right) = 18\csc^2\left(\frac{3\pi}{4}\right) = 18\left(\sqrt{2}\right)^2 = 36$$

Therefore, the value of $\dfrac{dy}{dx}$ **when** $x = \dfrac{\pi}{24}$ **is 36.**

5. Question Number: 21
Desc: Identify a point where a function is continuous but not differentiable

Consider the function $f(x) = \begin{cases} 2x+1 & \text{if } x \le 0 \\ -x^2+1 & \text{if } x > 0 \end{cases}$

Which of the following statements are true?
Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $f(0)$ exists

2. $\lim\limits_{x\to 0} f(x)$ does not exist

3. $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$ does not exist

4. $f(x)$ is continuous at $x = 0$
5. $f(x)$ is differentiable at $x = 0$

Solution:
$f(0) = 2(0) + 1 = 1$. So, $f(0)$ exists.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x+1) = 1 \text{ and } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(-x^2+1\right) = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \text{ implies } \lim_{x\to 0} f(x) \text{ exists.}$$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$ **implies $f(x)$ is continuous at $x = 0$.**

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(2h+1)-1}{h} = \lim_{h\to 0^-} \frac{2\cancel{h}}{\cancel{h}} = 2$$

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{-h^2+1-1}{h} = \lim_{h\to 0^+} (-h) = 0$$

$\lim\limits_{h\to 0^-} \dfrac{f(h)-f(0)}{h} \neq \lim\limits_{h\to 0^+} \dfrac{f(h)-f(0)}{h}$ **implies** $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$ **does not exist and $f(x)$ is not differentiable at $x = 0$.**

Therefore, the only correct statements are 1, 3 and 4 and the answer is 134.

Lesson topic: 4. Extreme Values and the Mean Value Theorem

1. Question Number: 22
Desc: Find the extreme values of a polynomial function over a closed interval

Consider the function $f(x) = 3x^3 + 9x^2$, where $x \in [-2,1]$. Let M be the absolute maximum of the function, and let m be the absolute minimum of the function. What whole number is equal to the difference $M - m$?

Solution:

$f'(x) = 9x^2 + 18x$

$f'(x) = 0$ **whenever** $9x^2 + 18x = 0$ **which occurs at** $x = -2$ **or** $x = 0$**.**

–2 and 0 are both in the domain of the function.
The absolute maximum and minimum of the function are the largest and the smallest numbers among *f*(–2), *f*(0), and *f*(1), respectively.

$f(-2) = 3(-2)^3 + 9(-2)^2 = -24 + 36 = 12$

$f(0) = 3(0)^3 + 9(0)^2 = 0$

$f(1) = 3(1)^3 + 9(1)^2 = 3 + 9 = 12$

So, M = 12 and m = 0.
Therefore, the whole number which is equal to the difference $M - m$ is 12.

2. Question Number: 23
Desc: Identify an interval where a trigonometric function satisfies the Rolle's theorem

Consider the function $f(x) = 4\sin\frac{x}{4}$ over the interval $[0, k\pi]$. What is the smallest positive value of k for which the function has at least one point c in the given interval where $f'(c) = 0$? Enter your answer as a whole number.

Solution:
By Rolle's theorem if $f(0) = f(k\pi) = 0$ and f is differentiable on $(0, k\pi)$ there exists a point $c \in (0, k\pi)$ for which $f'(c) = 0$.

Since the function $f(x) = 4\sin\frac{x}{4}$ is differentiable on $(0, k\pi)$ for any value of k, it remains to find such a value of k, for which $f(0) = f(k\pi) = 0$.

$$\mathbf{4\sin\frac{x}{4}=0 \quad\Leftrightarrow\quad \sin\frac{x}{4}=0 \quad\Leftrightarrow\quad \frac{x}{4}=\pi n,\ n\in\mathbb{Z} \quad\Leftrightarrow\quad x=4\pi n,\ n\in\mathbb{Z}}\ .$$

The positive values of *x* for which *f*(*x*) = 0 are 4π, 8π, 12π ...
Therefore, the smallest positive value of *k* is 4.

3. Question Number: 24
Desc: Identify an interval where a polynomial function satisfies the Rolle's theorem

Consider the function $f(x) = -2x^2 + 14x$ over the interval [0, *k*]. What is the value of *k* for which the function has at least one point *c* in the given interval where $f'(c) = 0$? Enter your answer as a whole number.

Solution:
By Rolle's theorem if *f*(0) = *f*(*k*) = 0 and *f* is differentiable on (0, *k*) there exists a point $c\in(0,k)$ for which *f'*(*c*) = 0.
Since the function $f(x) = -2x^2 + 14x$ is differentiable over the set of real numbers, it remains to find such a value of *k*, for which *f*(0) = *f*(*k*) = 0.
$f(x) = 0$ whenever $-2x^2 + 14x = -2x(x-7) = 0$ which occurs at *x* = 0 or *x* = 7.
Therefore, the value of *k* is 7.

4. Question Number: 25
Desc: Verify the necessary conditions for the Mean Value Theorem

Consider the function $f(x) = 2\tan\frac{x}{2}$.

Which of the following statements are true?
Select all that apply and enter the labels of the correct statements in an ascending order without spaces or separators.

1. $f(x)$ is continuous over the interval $(0,\ \pi)$.

2. $f(x)$ is differentiable over the interval $(0,\ \pi)$.

3. $f(x)$ is differentiable at $x = 0$.

4. $f(x)$ is continuous over the interval $[0,\ \pi]$.

5. There exists at least one value $c\in(0,\ \pi)$ such that $f'(c) = \frac{f(\pi)-f(0)}{\pi}$.

Solution:

Function *f*(*x*) is determined for all real values of *x* excluding the values for which $\cos\frac{x}{2}=0$.

$\cos\frac{x}{2}=0 \quad \Leftrightarrow \quad \frac{x}{2}=\frac{\pi}{2}+\pi n,\ n\in\mathbb{Z} \quad \Leftrightarrow \quad x=(1+2n)\pi,\ n\in\mathbb{Z}$

So, the function *f* is continuous and differentiable over the interval (0, π) and at *x* = 0.

$\lim\limits_{x\to\pi^-} f(x)=+\infty$ **and** $\lim\limits_{x\to\pi^+} f(x)=-\infty$ **implies *f* is not continuous at *x* = π and, consecutively, *f* is not continuous over the interval [0, π].**

Since the function $f(x)=2\tan\frac{x}{2}$ **is not continuous over the interval [0, π] the necessary condition for the Mean Value Theorem is not executed and there does not exist at least one value** $c\in(0,\pi)$ **such that** $f'(c)=\frac{f(\pi)-f(0)}{\pi}$**.**

Therefore, the only correct statements are 1, 2 and 3 and the answer is 123.

Lesson topic: 5. Curve Sketching

1. Question Number: 26
Desc: Find the second derivative of a function given parametrically

Consider the function given parametrically by $x=\sqrt{t}, y=t^2, t\geq 0$.

$\frac{d^2y}{dx^2}=kt$, where k is a whole number to be determined. What is the value of k?

Solution:,

$$\boldsymbol{\frac{d^2y}{dx^2}=\frac{y''(t)x'(t)-y'(t)x''(t)}{\left(x'(t)\right)^3}=\frac{(2t)'\frac{1}{2\sqrt{t}}+2t\cdot\frac{1}{4t\sqrt{t}}}{\left(\frac{1}{2\sqrt{t}}\right)^3}=8t\sqrt{t}\left(\frac{1}{\sqrt{t}}+\frac{1}{2\sqrt{t}}\right)=8t\sqrt{t}\cdot\frac{3}{2\sqrt{t}}=12t}$$

Therefore, the value of *k* is 12.

2. Question Number: 27
Desc: Recognize the graph of a function given parametrically

Consider the three functions given parametrically by

$f: x=0.75t, y=\sqrt{t}$ g: $x = t^2$, $y = t^9$ h: $x = -t^5$, $y = t$

Consider also the five graphs below labelled I, II, III, IV, and V.

Select all that apply and enter their labels in the same order as they appear (ascending order). Enter the labels without any spaces or commas.

1. The graph of *f* is the curve labelled I.

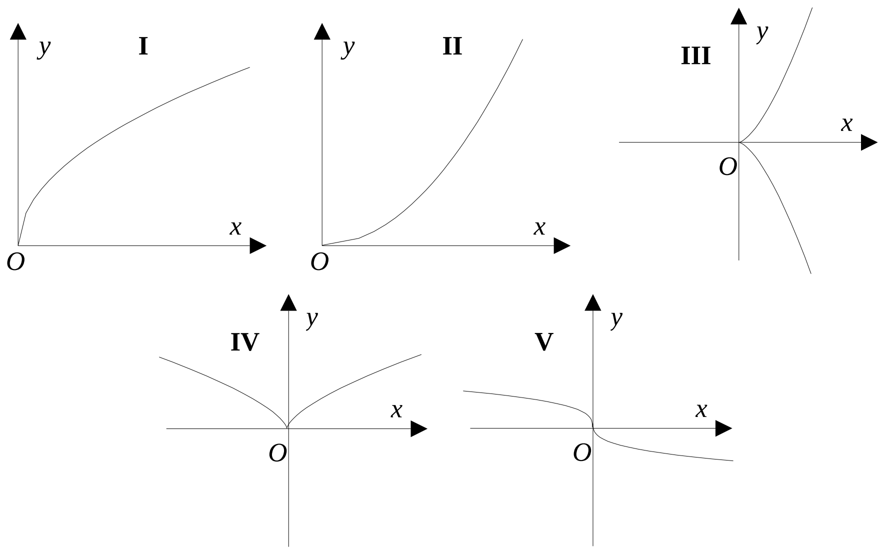
2. The graph of *f* is the curve labelled II.

3. The graph of *h* is the curve labelled V.

4. The graph of *h* is the curve labelled IV.

5. The graph of *g* is the curve labelled IV.

6. The graph of *g* is the curve labelled III.



Solution:

The graph of *f* is the curve labelled I (it is similar to the graph of $y=\sqrt{x}$), the graph of *g* is the curve labelled III (it is symmetric with respect to *x*-axis), and the graph of *h* is the curve labelled V. Therefore, the answer is 136.

Lesson topic: 6. Applications of the Derivative

1. Question Number: 28
Desc: Find the slope of the tangent to the graph of an implicit function

A function is defined implicitly by the equation $2x^3 + 2x + y^2 - 3y = 2$. In the xy-plane, the slope of the tangent to the graph of the function at the point (1, 1) is equal to k, where k is a constant. Find k, and enter its value as a whole number.

Solution:
To obtain the slope of the tangent to the curve $2x^3 + 2x + y^2 - 3y = 2$, differentiate both sides of implicit equation and solve the resulting expression for $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(2x^3 + 2x + y^2 - 3y\right) = \frac{d}{dx}(2)$$

$$6x^2 + 2 + 2y\frac{dy}{dx} - 3\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 3\frac{dy}{dx} = -\left(6x^2 + 2\right)$$

$$\frac{dy}{dx} = -\frac{6x^2 + 2}{2y - 3}$$

Substituting 1 for x and 1 for y in the expression for $\frac{dy}{dx}$ yields,

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{6(1)^2 + 2}{2(1) - 3} = 8.$$

Therefore, the value of k is 8.

2. Question Number: 29
Desc: Find the second derivative of an implicit function

Consider the implicit function $y^2 - x^2 = -64$. Derive a simplified expression for $\frac{d^2y}{dx^2}$ and find its value when $y = -2$. Enter this value as a whole number.

Solution:

Differentiating both sides of $y^2 - x^2 = -64$ yields, $2y\frac{dy}{dx} - 2x = 0$.

Solving for $\frac{dy}{dx}$ gives, $\frac{dy}{dx} = \frac{x}{y}$.

Differentiating both sides of $\frac{dy}{dx} = \frac{x}{y}$ yields, $\frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2}$.

Substituting $\frac{x}{y}$ for $\frac{dy}{dx}$ in the last equation gives,

$$\frac{d^2y}{dx^2} = \frac{y - \frac{x^2}{y}}{y^2} \text{ or } \frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}.$$

Substituting –64 for $y^2 - x^2$ and –2 for y in the expression for $\frac{d^2y}{dx^2}$ yields,

$$\frac{d^2y}{dx^2} = \frac{-64}{(-2)^3} = 8.$$

Therefore, the value of $\frac{d^2y}{dx^2}$ when $y = -2$ is 8.

3. Question Number: 30
Desc: Calculate related rates

The volume of a sphere is increasing at a constant rate of 5 cm^3/second. When the radius of the sphere is 5 cm, it increases at a rate of $\frac{1}{k\pi}$ cm/second, where k is a constant. Find k and enter its value as a whole number.

Solution:
The problem gives the rate of change of the volume V with respect to time as $\frac{dV}{dt} = 5 \text{ cm}^3/\text{sec}$ and asks for the rate of change of the radius of the sphere r with respect to time.

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} \text{ and } \frac{dV}{dr} = 4\pi r^2 \text{ imply } \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \text{ or } \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}.$$

Substituting $\frac{dV}{dt} = 5$ and $r = 5$ in the last expression gives,

$$\frac{dr}{dt} = \frac{\cancel{5}}{4\pi(5)^{\cancel{2}}} = \frac{1}{20\pi} \text{ cm/sec.}$$

Therefore, the value of *k* is 20.

Lesson topic: 7. Linearization

1. Question Number: 31
Desc: Find the linearization of a function

Consider the function *f* given by $y = f(x) = x^3 - 3x^2 + 2x$. Let *L*(*x*) be the linearization of *f* at *x* = 0.
What is the value of *L*(10)?

Solution: By definition, the expression
$L(x) = f(0) + f'(0)(x-0) = 0 + (3\cdot 0^2 - 6\cdot 0 + 2)x = 2x$ **is the linearization of *f* at *x* = 0.**
Then $L(10) = 2\cdot 10 = 20$.

2. Question Number: 32
Desc: Apply one step of Newton's method

<u>Sample question:</u>
Reference: Calculus I Chapter 4 Section 4

Question:
Consider the function *f* given by $y = f(x) = 2x^3 - 3x + 6$. Let $x_0 = 1$ be an initial estimate for a root of $f(x) = 0$. The next estimate for a root of $f(x) = 0$ using Newton's method is $x_1 = -\frac{k}{3}$, where *k* is a whole number to be determined. Give the value of *k*.

Solution: According to Newton's method, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{2\cdot 1^3 - 3\cdot 1 + 6}{6\cdot 1^2 - 3} = 1 - \frac{5}{3} = -\frac{2}{3}$.
Therefore, *k* = 2.

Lesson topic: 8. Integrals

1. Question Number: 33
Desc: Integrate the sum of algebraic and trigonometric functions

If $\int\left(\frac{36x^2}{\pi^3}-15\sin 3x\right)dx = I(x)+C$, find the value of $I(\pi)$.

Enter your answer as a whole number.

Solution:

$$\int\left(\frac{36x^2}{\pi^3}-15\sin 3x\right)dx=\int\frac{36}{\pi^3}x^2dx-\int 15\sin 3x dx=\frac{36}{\pi^3}\times\frac{x^3}{3}+15\times\frac{1}{3}\cos 3x+C$$

$$=\frac{12x^3}{\pi^3}+5\cos 3x+C$$

Thus, $I(x)=\frac{12x^3}{\pi^3}+5\cos 3x$ **.**

Substituting π for *x* in the expression for *I*(*x*) yields,

$$I(\pi)=\frac{12\cancel{\pi^3}}{\cancel{\pi^3}}+5\cos 3\pi=12+5(-1)=7\,.$$

Therefore, the value of $I(\pi)$ **is 7.**

2. Question Number: 34
Desc: Integrate a mixed function

If $\int(192x^2+8+3\pi\sec^2\pi x)\;dx = I(x)+C$, find the value of $I\left(\frac{1}{4}\right)$.

Enter your answer as a whole number.

Solution:

$$\int\left(192x^2+8+3\pi\sec^2\pi x\right)dx=\int 192x^2dx+\int 8dx+\int 3\pi\sec^2\pi x dx$$

$$=192\times\frac{x^3}{3}+8x+3\cancel{\pi}\times\frac{1}{\cancel{\pi}}\tan\pi x+C=64x^3+8x+3\tan\pi x+C$$

Thus, $I(x)=64x^3+8x+3\tan\pi x$ **.**

Substituting $\frac{1}{4}$ **for *x* in the expression for *I*(*x*) yields,**

$$I\left(\frac{1}{4}\right)=64\left(\frac{1}{4}\right)^3+8\left(\frac{1}{4}\right)+3\tan\frac{\pi}{4}=1+2+3=6.$$

Therefore, the value of $I\left(\frac{1}{4}\right)$ is 6.

3. Question Number: 35
Desc: Integrate a trigonometric function involving the sum of sine and cosine

If $\int(12\cos x-6\sin x)dx=I(x)+C$, find the value of $I(\theta)$, where $\cos\theta=\frac{4}{5}$ and $\sin\theta=\frac{3}{5}$.
Enter your answer as a whole number.

Solution:

$$\int(12\cos x-6\sin x)dx=\int 12\cos x dx-\int 6\sin x dx=12\sin x+6\cos x+C.$$

So, $I(x)=12\sin x+6\cos x$ and $I(\theta)=12\sin\theta+6\cos\theta$.

Substituting $\frac{4}{5}$ for cos θ and $\frac{3}{5}$ for sin θ in the expression for $I(\theta)$ yields,

$$I(\theta)=12\times\frac{3}{5}+6\times\frac{4}{5}=\frac{36+24}{5}=12.$$

Therefore, the value of $I(\theta)$ is 12.

4. Question Number: 36
Desc: Integrate an algebraic function involving the reciprocals of powers of x

If $\int\frac{2x-7x^2}{x^4}dx=I(x)+C$, find the value of $I\left(\frac{1}{3}\right)$.
Enter your answer as a whole number.

Solution:

$$\int\frac{2x-7x^2}{x^4}dx=\int\left(\frac{2x}{x^4}-\frac{7x^2}{x^4}\right)dx=\int\left(\frac{2\cancel{x}}{x^{\cancel{4}3}}-\frac{7\cancel{x^2}}{x^{\cancel{4}2}}\right)dx=\int\left(\frac{2}{x^3}-\frac{7}{x^2}\right)dx$$

$$=\int 2x^{-3}dx-\int 7x^{-2}dx=2\frac{x^{-2}}{-2}-7\frac{x^{-1}}{-1}+C=-\frac{1}{x^2}+\frac{7}{x}+C.$$

Thus, $I(x)=-\frac{1}{x^2}+\frac{7}{x}$.

Substituting $\frac{1}{3}$ for x in the expression for $I(x)$ yields,

$$I\left(\frac{1}{3}\right)=-\frac{1}{\left(\frac{1}{3}\right)^2}+\frac{7}{\frac{1}{3}}=-9+21=12.$$

Therefore, the value of $I\left(\frac{1}{3}\right)$ **is 12.**

5. Question Number: 37
Desc: Use substitution to integrate a radical function

If $\int x^2\sqrt{x^3+9}\ dx = I(x)+C$, find the value of $I(3)$.
Enter your answer as a whole number.

Solution:
Letting $u = x^3 + 9$ **gives** $du = 3x^2dx$**. Substituting in the integrand yields,**

$$\int x^2\sqrt{x^3+9}dx=\int\frac{1}{3}\sqrt{x^3+9}\left(3x^2\right)dx=\int\frac{1}{3}\sqrt{u}du=\int\frac{1}{3}u^{\frac{1}{2}}du$$

$$=\frac{1}{3}\times\frac{2}{3}u^{\frac{3}{2}}+C=\frac{2}{9}u\sqrt{u}+C=\frac{2}{9}\left(x^3+9\right)\sqrt{x^3+9}+C.$$

Thus, $I(x)=\frac{2}{9}\left(x^3+9\right)\sqrt{x^3+9}$.

Substituting 3 for x **in the expression for** $I(x)$ **yields,**

$$I(3)=\frac{2}{9}\left(3^3+9\right)\sqrt{3^3+9}=\frac{2}{9}\times 36\times 6=48.$$

Therefore, the value of $I(3)$ **is 48.**

6. Question Number: 38
Desc: Use substitution to integrate a power of tangent

If $\int 21\tan^2 x\sec^2 x\ dx = I(x)+C$, find the value of $I\left(\frac{\pi}{4}\right)$.

Enter your answer as a whole number.

Solution:
Letting $u = \tan x$ **gives** $du = \sec^2 xdx$**. Substituting in the integrand yields,**

$$\int 21\tan^2 x\sec^2 xdx=\int 21u^2du=21\times\frac{u^3}{3}+C=7u^3+C=7\tan^3 x+C.$$

Thus, $I(x)=7\tan^3 x$.

Substituting $\frac{\pi}{4}$ for x in the expression for $I(x)$ yields,

$$I\left(\frac{\pi}{4}\right) = 7\tan^3\frac{\pi}{4} = 7\times 1 = 7.$$

Therefore, the value of $I\left(\frac{\pi}{4}\right)$ is 7.

7. Question Number: 39
Desc: Integrate a power of a linear expression

If $\int \frac{2}{\sqrt[3]{x+25}}\,dx = I(x)+C$, find the value of $I(100)$.
Enter your answer as a whole number.

Solution:

$$\int \frac{2}{\sqrt[3]{x+25}}dx = \int 2\left(x+25\right)^{-\frac{1}{3}}dx = 2\frac{\left(x+25\right)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \cancel{2}\times\frac{3}{\cancel{2}}\left(x+25\right)^{\frac{2}{3}} + C = 3\left(\sqrt[3]{x+25}\right)^2 + C$$

Thus, $I\left(x\right) = 3\left(\sqrt[3]{x+25}\right)^2$.
Substituting 100 for x in the expression for $I(x)$ yields,

$$I\left(100\right) = 3\left(\sqrt[3]{100+25}\right)^2 = 3\left(5\right)^2 = 75.$$

Therefore, the value of $I\left(100\right)$ is 75.

8. Question Number: 40
Desc: Use substitution to integrate a power of sine

If $\int 320\cos x\sin^3 x\,dx = I(x)+C$, find the value of $I\left(\frac{\pi}{6}\right)$.

Enter your answer as a whole number.

Solution:
Letting $u = \sin x$ gives $du = \cos x dx$. Substituting in the integrand yields,

$$\int 320\cos x\sin^3 x dx = \int 320\sin^3 x\cos x dx = \int 320u^3 du = 320\times\frac{u^4}{4} + C = 80\sin^4 x + C.$$

Thus, $I\left(x\right) = 80\sin^4 x$.

Substituting $\frac{\pi}{6}$ for x in the expression for $I(x)$ yields,

$$I\left(\frac{\pi}{6}\right) = 80\sin^4\frac{\pi}{6} = 80\left(\frac{1}{2}\right)^4 = 80\times\frac{1}{16} = 5.$$

Therefore, the value of $I\left(\frac{\pi}{6}\right)$ is 5.

9. Question Number: 41
Desc: Integrate a trigonometric function involving the sum of tangent and secant

If $\int \sec x(8\sqrt{2}\tan x - 7\sec x)dx = I(x) + C$, find the value of $I\left(\frac{\pi}{4}\right)$.

Enter your answer as a whole number.

Solution:

$$\int \sec x\left(8\sqrt{2}\tan x - 7\sec x\right)dx = \int 8\sqrt{2}\sec x\tan x dx - \int 7\sec^2 x dx = 8\sqrt{2}\sec x - 7\tan^2 x + C$$

So, $I(x) = 8\sqrt{2}\sec x - 7\tan^2 x$.

Substituting $\frac{\pi}{4}$ for x in the expression for $I(x)$ yields,

$$I\left(\frac{\pi}{4}\right) = 8\sqrt{2}\sec\frac{\pi}{4} - 7\tan^2\frac{\pi}{4} = 8\sqrt{2}\times\sqrt{2} - 7\times 1 = 16 - 7 = 9.$$

Therefore, the value of $I\left(\frac{\pi}{4}\right)$ is 9.

Lesson topic: 9. The Fundamental Theorems of Calculus

1. Question Number: 42
Desc: Locate the point at which the value of a function equals its average

If $f(x) = x^2 - 2$, the point $c \in (0,2)$ for which $f(c) = f_{av}$ *(average value)* is the solution of the equation $c^2 = \frac{k}{3}$, where *k* is a constant. Find *k*, and enter your answer as a whole number.

Solution:
The average value of $f(x) = x^2 - 2$ on the interval (0, 2) is given by

$$f_{av} = \frac{1}{2-0}\int_0^2 \left(x^2 - 2\right)dx = \frac{1}{2}\int_0^2 x^2 dx - \int_0^2 dx = \frac{1}{2}\left(\frac{2^3}{3} - 0\right) - \left(2-0\right) = \frac{4}{3} - 2 = -\frac{2}{3}.$$

Hence, $f\left(c\right) = c^2 - 2 = -\frac{2}{3}$ **or** $c^2 = \frac{4}{3}$.
Therefore, the value of *k* is 4.

2. Question Number: 43
Desc: Use substitution to evaluate the definite integral of a rational function

Evaluate $\int_0^3 3x^2\sqrt{x^3+9}\,dx$.

Solution:
Letting $u = x^3 + 9$ gives $du = 3x^2dx$. When $x = 0$, $u = 9$ and when $x = 3$, $u = 36$.

$$\int_0^3 3x^2\sqrt{x^3+9}dx = \int_9^{36}\sqrt{u}du = \int_9^{36} u^{\frac{1}{2}}du = \frac{2}{3}\left(\sqrt{u}\right)^3\Big|_9^{36} = \frac{2}{3}\left(6^3 - 3^3\right) = 126.$$

Therefore, $\int_0^3 3x^2\sqrt{x^3+9}\,dx = 126.$

3. Question Number: 44
Desc: Use substitution to find the definite integral of a power of cosine

If $\int_0^{\frac{\pi}{10}} \sin 5x \cos^2 5x\, dx = \frac{1}{k}$, find *k* and enter your answer as a whole number.

Solution:

Letting $u = \cos 5x$ gives $du = -5\sin 5x dx$. When $x = 0$, $u = 1$ and when $x = \frac{\pi}{10}$, $u = 0$.

Substituting in the integrand yields,

$$\int_0^{\frac{\pi}{10}} \sin 5x \cos^2 5x dx = \int_0^{\frac{\pi}{10}} \left(-\frac{1}{5}\right) \cos^2 5x (-5) \sin 5x dx = -\frac{1}{5}\int_1^0 u^2 du = -\frac{1}{5} \times \frac{u^3}{3}\Bigg|_1^0 = \frac{1}{15}.$$

Therefore, the value of *k* is 15.

4. Question Number: 45
Desc: Apply the first fundamental theorem of calculus where the integrand is the product of two functions

If $f(x) = \int_x^{x^3} 5\theta \cos \pi\theta \, d\theta$, what is the value of $f'(-1)$? Enter your answer as a whole number.

Solution:

$$f(x) = \int_x^{x^3} 5\theta \cos \pi\theta d\theta = \int_x^1 5\theta \cos \pi\theta d\theta + \int_1^{x^3} 5\theta \cos \pi\theta d\theta = -\int_1^x 5\theta \cos \pi\theta d\theta + \int_1^{x^3} 5\theta \cos \pi\theta d\theta$$

$$f'(x) = -5x \cos \pi x + (3x^2) 5x \cos \pi x = -5x \cos \pi x + 15x^3 \cos \pi x$$

Substituting –1 for *x* in the expression for $f'(x)$ yields,

$$f'(-1) = -5(-1)\cos(-\pi) + 15(-1)^3 \cos(-\pi) = 5(-1) - 15(-1) = -5 + 15 = 10.$$

Therefore, the value of $f'(-1)$ is 10.

5. Question Number: 46
Desc: Apply the first fundamental theorem of calculus where the integrand is a single function

If $f(x) = \int_x^{5x} 15\cos^2 w \, dw$, evaluate $f'(\pi)$. Enter your answer as a whole number.

Solution:

$$f(x) = \int_x^{5x} 15\cos^2 w dw = \int_x^1 15\cos^2 w dw + \int_1^{5x} 15\cos^2 w dw = -\int_1^x 15\cos^2 w dw + \int_1^{5x} 15\cos^2 w dw$$

$$f'(x) = -15\cos^2 x + 15 \times 5 \times \cos^2 5x = -15\cos^2 x + 75\cos^2 5x$$

Substituting π for *x* in the expression for $f'(x)$ yields,

$$f'(\pi) = -15\cos^2 \pi + 75\cos^2 5\pi = -15(1) + 75(1) = 60.$$

Therefore, the value of $f'(\pi)$ is 60.

Lesson topic: 10. Applications of the Integral

1. Question Number: 47
Desc: Find the volume of a solid of revolution by revolving about the x-axis

The region bounded by the curve $y = x^2\sqrt{x}$ and the lines $y = \sqrt{2}x$ and $y = 1$ is rotated completely about the x-axis. The volume of the solid generated is $\frac{\pi}{k}$, where k is a whole number to be determined. Give the value of k.

Solution: The volume of the solid generated is

$$V = \int_a^b \pi\left\{\left[f(x)\right]^2 - \left[g(x)\right]^2\right\}dx = \int_0^1 \pi\left\{2x^2 - x^5\right\}dx = \pi\left(\frac{2}{3}x^3 - \frac{x^6}{6}\right)\Bigg|_0^1 = \pi\left(\frac{2}{3} - \frac{1}{6}\right) = \frac{\pi}{2}.$$

Hence, k = 2.

2. Question Number: 48
Desc: Find the volume generated by the rotation of the area under the graph of x^n, n>0, about the x-axis

The region in the xy-plane above the x-axis and bounded by the graphs of $y = \frac{5}{4}\sqrt[3]{x}$ and $x = 8$ is rotated a full rotation about the x-axis to generate a solid with volume equal to $k\pi$, where k is a constant. Find k, and enter your answer as a whole number.

Solution:
The curve cuts the x-axis at $x = 0$. Hence, the region that is rotated about the x-axis extend from $x = 0$ to $x = 8$.

$$V = \int_a^b \pi y^2 dx = \int_0^8 \pi\left(\frac{5}{4}\sqrt[3]{x}\right)^2 dx = \frac{25\pi}{16}\int_0^8 x^{\frac{2}{3}}dx = \frac{25\pi}{16}\times\frac{3}{5}x^{\frac{5}{3}}\Bigg|_0^8 = \frac{\overset{5}{\cancel{25}}\pi}{\cancel{16}}\times\frac{3}{\cancel{5}}\times\cancel{32}^{2} = 30\pi$$

Therefore, the value of k is 30.

3. Question Number: 49
Desc: Find the volume generated by the rotation of the area under the graph of x^n, n<0, about the x-axis

The volume of the solid generated by the rotation of the area under the graph of the function $y = \frac{1}{\sqrt{x^3}}, x \in [2,3]$ a full rotation about the x-axis is equal to $\frac{5\pi}{k}$, where k is a constant. Find k, and enter your answer as a whole number.

Solution:

$$\boldsymbol{V = \int_a^b \pi y^2 dx = \int_2^3 \pi\left(\frac{1}{\sqrt{x^3}}\right)^2 dx = \pi\int_2^3 x^{-3}dx = \pi \times \frac{x^{-2}}{-2}\bigg|_1^2 = -\frac{\pi}{2x^2}\bigg|_2^3 = -\frac{\pi}{2}\left(\frac{1}{9}-\frac{1}{4}\right) = \frac{5\pi}{72}}$$

Therefore, the value of *k* is 72.

4. Question Number: 50
Desc: Find the volume of a solid having the x-axis as an axis of symmetry

A solid is bounded by two planes perpendicular to the x-axis at $x = -2$ and $x = 2$. The cross section of the solid with a plane perpendicular to the x-axis is a circle whose diameter in the xy-plane is bounded by the graphs of $y = x^2 - 4$ and $y = 4 - x^2$, where $x \in [-2, 2]$. The volume of this solid is equal to $\frac{k\pi}{15}$, where k is a constant. Find k, and enter your answer as a whole number.

Solution:

The radius of the cross-section is $\boldsymbol{R = \frac{1}{2}\left(4 - x^2 - x^2 + 4\right) = 4 - x^2}$**.**

$$\boldsymbol{A(x) = \pi R^2 = \pi\left(4 - x^2\right)^2}$$

The volume of the solid is:

$$\boldsymbol{V = \int_a^b A(x)dx = \int_{-2}^2 \pi\left(4 - x^2\right)^2 dx = \pi\int_{-2}^2 \left(16 - 8x^2 + x^4\right)dx = \pi\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\bigg|_{-2}^2}$$

$$\boldsymbol{= \pi\left[\left(16(2) - \frac{8(2)^3}{3} + \frac{2^5}{5}\right) - \left(16(-2) - \frac{8(-2)^3}{3} + \frac{(-2)^5}{5}\right)\right] = \pi\left(64 - \frac{128}{3} + \frac{64}{5}\right)}$$

$$\boldsymbol{= 64\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 64\pi \times \frac{8}{15} = \frac{512\pi}{15}}$$

Therefore, the value of *k* is 512.

Lesson topic: 11. Lengths of Smooth Curves

1. Question Number: 51
Desc: Find the length of a curve described by an explicit equation

The length of the curve with equation $x = \frac{2y\sqrt{y}}{3}$ between $y = 0$ and $y = 6$ is equal to $\frac{k}{3}$, where k is a constant. Find k and enter your answer as a whole number.

Solution:

$$x' = \left(\frac{4}{3}y^{\frac{3}{2}}\right)' = \frac{4}{3}\times\frac{3}{2}y^{\frac{1}{2}} = 2\sqrt{y}$$

$$L = \int_0^6 \sqrt{1+\left(2\sqrt{y}\right)^2}\,dy = \int_0^6 \sqrt{1+4y}\,dy$$

Let $u = 1 + 4y$, $du = 4dy$. Also, when $y = 0$, $u = 1$ and when $y = 6$, $u = 25$.

$$L = \frac{1}{4}\int_1^{25}\sqrt{u}\,du = \frac{1}{4}\int_1^{25} u^{\frac{1}{2}}du = \frac{1}{{}_2\cancel{4}}\times\frac{\cancel{2}}{3}u^{\frac{3}{2}}\Big|_1^{25} = \frac{1}{6}\left(25^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{124}{6} = \frac{62}{3}$$

Therefore, the value of k is 62.

2. Question Number: 52
Desc: Find the length of a curve described by an integral

The length of the graph of the function $f(x) = \int_1^x \sqrt{9u^2 - 1}\,du$ between $x = 2$ and $x = 5$ is equal to $\frac{k}{2}$, where k is a constant. Find k and enter your answer as a whole number.

Solution:

The first fundamental theorem of calculus gives, $f'(x) = \sqrt{9x^2 - 1}$ and the length of the curve is:

$$L = \int_2^5 \sqrt{1+\left(\sqrt{9x^2-1}\right)^2}\,dx = \int_2^5 3x\,dx = \frac{3x^2}{2}\Big|_2^5 = \frac{3(5)^2}{2} - \frac{3(2)^2}{2} = \frac{63}{2}.$$

Therefore, the value of k is 63.

Lesson topic: 12. The Natural Logarithm

1. Question Number: 53
Desc: Find the definite integral of f'(x)/f(x)

The result of the integral $\int_0^2 \frac{6x^2}{x^3+2}dx$ may be reduced to the form ln *k*, where *k* is a constant. Find *k* and enter its value as a whole number.

Solution:
Let $u = x^3 + 2$ then du = $3x^2dx$. Also, $u(0) = 2$ and $u(2) = 10$.

$$L = \int_0^2 \frac{6x^2}{x^3+1}dx = 2\int_2^{10} \frac{1}{u}du = 2\ln|u|\Big|_2^{10} = 2(\ln 10 - \ln 2) = 2\ln\frac{10}{2} = \ln 5^2 = \ln 25$$

Therefore, the value of *k* is 25.

Lesson topic: 13. The Exponential Function

1. Question Number: 54
Desc: Use substitution to find the definite integral of an exponential function

If $\int\left(\frac{6e^{-\cot y}}{\sin^2 y}\right)dy = I(y)+C$, what is the value of $I\left(\frac{\pi}{2}\right)$. Enter your answer as a whole number.

Solution:

Letting $u = -\cot y$ gives $du = \frac{dy}{\sin^2 y}$. Substituting in the integrand yields,

$$\int\left(\frac{6e^{-\cot y}}{\sin^2 y}\right)dy = \int 6e^u du = 6e^u + C = 6e^{-\cot y} + C.$$

Thus, $I(y) = 6e^{-\cot y}$.

Substituting $\frac{\pi}{2}$ for y in the expression for $I(y)$ yields,

$$I\left(\frac{\pi}{2}\right) = 6e^{-\cot\frac{\pi}{2}} = 6e^0 = 6.$$

Therefore, the value of $I\left(\frac{\pi}{2}\right)$ is 6.

2. Question Number: 55
Desc: Find the definite integral of f'(x)/f(x) where f(x) is an exponential function

The result of the integral $\int_0^{\ln 10}\frac{3e^x}{e^x+2}dx$ may be reduced to the form ln *k*, where *k* is a constant. Find *k* and enter its value as a whole number.

Solution:
Letting $u = e^x + 2$ gives $du = e^x dx$. Also, $u(0) = 3$ and $u(\ln 10) = 12$.
Substituting in the integrand yields,

$$\int_0^{\ln 10}\frac{3e^x}{e^x+2}dx = \int_3^{12}\frac{3}{u}du = 3\ln|u|\Big|_3^{12} = 3(\ln 12 - \ln 3) = 3\ln\frac{12}{3} = \ln 4^3 = \ln 64.$$

Therefore, the value of *k* is 64.

3. Question Number: 56
Desc: Find the definite integral of an exponential function

If $\int_{\ln 4}^{\ln 8} 3e^x dx = a$, what is the value of *a*? Enter your answer as a whole number.

Solution:

$$\int_{\ln 4}^{\ln 8} 3e^x dx = 3e^x\Big|_{\ln 4}^{\ln 8} = 3\left(e^{\ln 8} - e^{\ln 4}\right) = 3(8-4) = 12$$

Therefore, the value of *a* is 12

4. Question Number: 57
Desc: Use substitution to find the definite integral of an exponential function

The result of the integral $\int_0^2 5x^4 e^{x^5} dx$ can be reduced to the form $e^a - b$, where *a* and *b* are constants. Find the value of *a* + *b* and enter your answer as a whole number.

Solution:
Let $u = x^5$ then $du = 5x^4dx$. Also, $u(0) = 0$ and $u(2) = 32$.

$$\int_0^2 5x^4 e^{x^5} dx = \int_0^{32} e^u du = e^u\Big|_0^{32} = e^{32} - 1$$

The respective values of *a* and *b* are 32 and 1. Therefore, *a* + *b* = 33.

5. Question Number: 58
Desc: Find the derivative of a logarithmic function

If $\int 3^x dx = f(x) + C$, find $f(3)$. Use $\frac{9}{8}$ as an approximate value for ln 3 and enter your answer as a whole number.

Solution:

$$\int 3^x dx = \frac{3^x}{\ln 3} + C$$

Hence, $f(x) = \frac{3^x}{\ln 3}$.

Substituting 3 for *x* and using $\frac{9}{8}$ as the approximate value for ln 3 in the expression for *f*(*x*) yields,

$$f(3) = \frac{3^3}{\frac{9}{8}} = \cancel{27}^{3} \times \frac{8}{\cancel{9}} = 24.$$

Therefore, *f*(3) = 24.

6. Question Number: 59
Desc: Find the derivative of a logarithmic function

If $f(x) = 7\log_{e^2}(2x^4 - 4x^2 + 6)$, find $f'(1)$. Enter your answer as a whole number.

Solution:

$$f(x) = 7\log_{e^2}\left(2x^4 - x^2 + 6\right) = \frac{7}{2}\log_e\left(2x^4 - x^2 + 6\right) = \frac{7}{2}\ln\left(2x^4 - x^2 + 6\right)$$

$$f'(x) = \left[\frac{7}{2}\ln\left(2x^4 - x^2 + 6\right)\right]' = \frac{7}{2}\times\frac{\left(2x^4 - x^2 + 6\right)'}{2x^4 - x^2 + 6} = \frac{7}{2}\times\frac{8x^3 - 2x}{2x^4 - x^2 + 6}$$

Substituting 1 for *x* in the expression for $f'(x)$ yields,

$$f'(1) = \frac{7}{2}\times\frac{8(1)^3 - 2(1)}{2(1)^4 - 1^2 + 6} = \frac{\cancel{7}}{\cancel{2}}\times\frac{\cancel{6}^{\,3}}{\cancel{7}} = 3.$$

Therefore, $f'(1) = 3$.

Lesson topic: 14. Inverse Trigonometric Functions

1. Question Number: 60
Desc: Find the derivative of a function which involves arcos and study its properties

Consider the function *f* given by $y = f\left(x\right) = 2\cos^{-1} e^{\frac{x}{2}}$.
Which of the following is true about the derivative of *f*?

Select all that apply and enter their labels in the same order as they appear (ascending order). Enter the labels without any spaces or commas.

1. Its rule is $y' = \dfrac{-e^{x/2}}{\sqrt{1-e^{x}}}$.

2. Its rule is $y' = \dfrac{e^{x/2}}{\sqrt{1-e^{x}}}$.

3. Its rule is $y' = -\dfrac{e^{x/2}}{\sqrt{1-e^{x/2}}}$.

4. It is defined for all $-2 \le x \le 2$.

5. It is defined for all $x < 0$.

Solution: $y' = \left(2\cos^{-1} e^{\frac{x}{2}}\right)' = 2 \cdot \dfrac{-1}{\sqrt{1-\left(e^{\frac{x}{2}}\right)^{2}}} \cdot e^{\frac{x}{2}} \cdot \dfrac{1}{2} = \dfrac{-e^{x/2}}{\sqrt{1-e^{x}}}$.

It is defined for all x such that $1-e^{x} > 0 \Leftrightarrow e^{x} < 1 \Leftrightarrow x < 0$.
So the answer is 15.

2. Question Number: 61
Desc: Find the derivative of an inverse tan

If $f(x) = \dfrac{1}{12}\tan^{-1} x^{3}$, then $f'(2) = \dfrac{1}{k}$, where *k* is a constant. Enter the value of *k* as a whole number.

Solution:

$$f'\left(x\right) = \left(\frac{1}{12}\tan^{-1} x^{3}\right)' = \frac{1}{12} \times \frac{1}{1+\left(x^{3}\right)^{2}} \times \left(x^{3}\right)' = \frac{1}{\cancel{12}_{4}} \times \frac{\cancel{3}x^{2}}{1+x^{6}} = \frac{x^{2}}{4\left(1+x^{6}\right)}$$

Substituting 2 for *x* in the expression for $f'(x)$ yields,

$$\boldsymbol{f'(2)=\frac{2^2}{4(1+2^6)}=\frac{1}{65}}.$$

Therefore, the value of *k* is 65.

3. Question Number: 62
Desc: Find a definite integral resulting an inverse tan

Which of the following is equal to the value of the integral $\int_0^1 \frac{6x^2}{1+x^6}dx$?

Enter the labels of the correct answers in an ascending order.

1. $2\tan^{-1} 1$

2. $\tan^{-1}\frac{1}{2}$

3. **2**π

4. $\frac{\pi}{2}$

5. $\frac{1}{2}$

6. 1

Solution:
Letting *u* = x^3 gives *du* = $3x^2dx$. Also, *u*(0) = 0 and *u*(1) = 1.

Substituting in the integrand yields,

$$\boldsymbol{\int_0^1 \frac{6x^2}{1+x^6}dx = 2\int_0^1 \frac{1}{1+u^2}du = 2\left(\tan^{-1} u\right)\Big|_0^1 = 2\left(\tan^{-1}1-\tan^{-1}0\right)=2\tan^{-1}1=2\times\frac{\pi}{4}=\frac{\pi}{2}}.$$

Therefore, the only correct options are 1 and 4 and the answer is 1.

4. Question Number: 63
Desc: Find the integral of a function where the result involves arcsine

$\int_0^{1/4} \frac{1}{\sqrt{1-16x^2}}\,dx=\frac{\pi}{k}$, where *k* is whole numbers to be determined. What is the value of *k*?

Solution:

$$\int_0^{1/4} \frac{1}{\sqrt{1-16x^2}}\,dx = \frac{1}{4}\int_0^{1/4} \frac{1}{\sqrt{1-(4x)^2}}\,d(4x) = \frac{1}{4}\arcsin(4x)\Big|_0^{1/4} = \frac{1}{4}\arcsin(1) - \frac{1}{4}\arcsin(0) =$$

$$\frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

Therefore, k = 8.

5. Question Number: 64
Desc: Find the integral of a function where the result involves arcsine

$\int \frac{dx}{\sqrt{-143+24x-x^2}} = \sin^{-1}\left(\frac{ax+b}{d}\right) + C$, where a, b, and d are whole numbers to be determined and C is the constant of integration. What is the value of $\frac{ax+b}{d}$ when x = 100?

(Ignore the fact whether the original integrand is defined for x = 100 or not.)

Solution: $\int \frac{dx}{\sqrt{-143+24x-x^2}} = \int \frac{dx}{\sqrt{1-(x-12)^2}} = \arcsin(x-12)+C$

So $\frac{ax+b}{d} = x-12$ and when x = 100 then $\frac{ax+b}{d} = 100-12 = 88$.

6. Question Number: 65
Desc: Find the derivative of a function involving arcsine

If $y = 8\sin^{-1}\left(\frac{5}{x}\right)$ then $y'(13) = y'\big|_{x=13} = \frac{-k}{39}$, where k is a whole number to be determined. Give the value of k.

Solution: $y' = \left(8\sin^{-1}\left(\frac{5}{x}\right)\right)' = 8\cdot\frac{1}{\sqrt{1-\frac{25}{x^2}}}\cdot\frac{-5}{x^2} = -\frac{40}{x\sqrt{x^2-25}}$

$$y'\big|_{x=13} = -\frac{40}{13\sqrt{169-25}} = -\frac{40}{13\cdot 12} = -\frac{10}{39}$$

So k = 10.

7. Question Number: 66
Desc: Find the integral of a function where the result involves arctan

$\int \frac{dx}{81x^2+18x+5} = \frac{1}{a}\tan^{-1}\left(\frac{bx+1}{d}\right)+C$, where *a*, *b*, and *d* are whole numbers to be determined and *C* is the constant of integration. What is the value of $\frac{bx+1}{d}$ when *x* = 3?

Hint: Determine *b* and *d* and substitute 3 for *x* in the expression $\frac{bx+1}{d}$, and simplify. The answer is a whole number.

Solution: $\int \frac{dx}{81x^2+18x+5} = \int \frac{dx}{(9x+1)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{9x+1}{2}\right)^2+1} = \frac{1}{4}\cdot\frac{2}{9}\int \frac{d\left(\frac{9x+1}{2}\right)}{\left(\frac{9x+1}{2}\right)^2+1} =$

$\frac{1}{18}\arctan\left(\frac{9x+1}{2}\right)$

Here $\frac{bx+1}{d} = \frac{9x+1}{2}$**. When** x **= 3 then** $\frac{bx+1}{d} = \frac{9\cdot 3+1}{2} = 14$**.**

Lesson topic: 15. Limits of Functions

1. Question Number: 67
Desc: Evaluate a definite integral involving an inverse sine

If $\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x = a$, which of the following is true about the value of a? Select all that apply and enter the labels of the correct choices in an ascending order without spaces or separators.

1. $7 < a < 8$

2. $4 < a < 5$

3. $0 < a < 1$

4. $a = e^{-2}$

5. $a = e^2$

6. $a = 0$
7. $a = 1$

Solution:

$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x$ **has the form** 1^∞.

Let $y=\left(1-\frac{2}{x}\right)^x$ **then** $\ln y = x\ln\left(1-\frac{2}{x}\right)=\frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}}$ **and** $\lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}}$ **has the form** $\frac{0}{0}$.

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{1}{1-\frac{2}{x}}\times\frac{2}{\cancel{x^2}}}{-\frac{1}{\cancel{x^2}}}=-\lim_{x\to\infty}\frac{2}{1-\frac{2}{x}}=-2.$$

Hence, $\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x=\lim_{x\to\infty}y=e^{-2}$.

The value of e^{-2} **is less than 1.**
Therefore, the only correct options are 3 and 4 and the answer is 34.

Lesson topic: 16. Differential Equations

1. Question Number: 68
Desc: Solve a variable separable differential equation involving arcsine

Consider the differential equation $\frac{dy}{dx} = 14x\sqrt{25 - y^2}$, $y = 0$ when $x = 0$.
The particular solution of this equation $y = a\sin bx^2$, where a and b are constants to be determined.
Give only the value of b.

Solution:

$$\frac{dy}{dx} = 14x\sqrt{25 - y^2} \Leftrightarrow \frac{dy}{\sqrt{25 - y^2}} = 14x\,dx \Leftrightarrow \frac{d\left(\frac{y}{5}\right)}{\sqrt{1 - \left(\frac{y}{5}\right)^2}} = 14x\,dx \Leftrightarrow \arcsin\left(\frac{y}{5}\right) = 7x^2 + C$$

$$\Leftrightarrow y = 5\sin\left(7x^2 + C\right)$$

$y = 0$ when $x = 0$, so $C = 0$. Therefore, $y = 5\sin\left(7x^2\right)$ and $b = 7$.

2. Question Number: 69
Desc: Find the integrating factor of a linear differential equation

The integrating factor for $xy' = 5y - x^7$ is ax^b, where a and b are constants to be determined.
Give the value of the integrating factor that corresponds to $x = \frac{1}{3}$.

Solution: $xy' = 5y - x^7 \Leftrightarrow y' - \frac{5}{x}y = -x^6$

An integrating factor equals $\nu(x) = e^{-\int \frac{5}{x}dx} = e^{-5\ln x} = x^{-5}$.

Therefore, the integrating factor that corresponds to $x = \frac{1}{3}$ is $\left(\frac{1}{3}\right)^{-5} = 3^5 = 243$.

3. Question Number: 70
Desc: Solve a linear differential equation

Consider the differential equation $xy' + 4y = x^2$, $y = \frac{7}{6}$ when $x = 1$.

The value of y when $x = 2$ is $\frac{k}{48}$. What is the value of k?

Hint: Find the particular solution, substitute 2, and then simplify.

Solution: $xy' + 4y = x^2 \Leftrightarrow y' + \frac{4}{x}y = x$

An integrating factor equals $\nu(x) = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$.

It yields, $x^4y' + 4x^3y = x^5 \Leftrightarrow x^4y = \frac{x^6}{6} + C$.

As $y = \frac{7}{6}$ **when** $x = 1$**, we obtain that** $C = 1$**.**

Then the value of y **when** $x = 2$ **is** $y = \frac{\frac{2^6}{6} + 1}{2^4} = \frac{2}{3} + \frac{1}{16} = \frac{35}{48}$.

4. Question Number: 71
Desc: Solve a variable separable differential equation

Consider the differential equation $\frac{dy}{dx} = \frac{x}{3y}$, $y = -1$ when $x = -1$.

The particular solution of this equation has the form $-\sqrt{A + Bx^2}$, where A and B are constants to be determined. The value of $A + Bx^2$ when $x = \frac{1}{4}$ is $\frac{k}{48}$. What is the value of k?

Solution: $\frac{dy}{dx} = \frac{x}{3y} \Leftrightarrow 3ydy = xdx \Leftrightarrow \frac{3}{2}y^2 = \frac{x^2}{2} + C \Leftrightarrow 3y^2 = x^2 + 2C$

$y = -1$ **when** $x = -1 \Leftrightarrow C = 1$

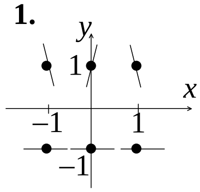
Hence, $y = -\sqrt{\frac{x^2 + 2C}{3}} = -\sqrt{\frac{x^2 + 2}{3}}$**. If** $x = \frac{1}{4}$ **then** $\frac{x^2 + 2}{3} = \frac{\frac{1}{16} + 2}{3} = \frac{33}{48}$**, so** $k = 33$**.**

5. Question Number: 72
Desc: Recognize the slope field of a differential equation

Which of the following represent a slope field for the differential equation $\frac{dy}{dx} = (y+1)^2 \cos \pi x$ at the six indicated points?

Select all that apply and enter their labels in the same order as they appear (ascending order). Enter the labels without any spaces or commas.

1.



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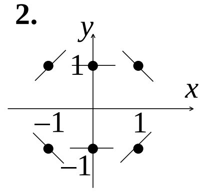

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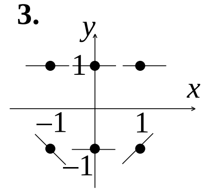

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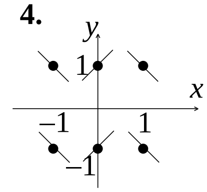
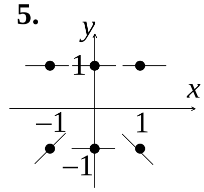


5.



Solution:
Substituting pairs (–1, 1), (0, 1), (1, 1), (–1, –1), (0, –1), (1, –1) into the given differential equation, we find that the slopes in these points are –4, 4, –4, 0, 0, 0. The first graph corresponds to it. Therefore, the answer is 1.

Lesson topic: 17. Trigonometic Expressions

1. Question Number: 73
Desc: Integrate the cube of a sine

If $\int \frac{1}{2}\cos^3 2x dx = I(x) + C$, then $I\left(\frac{\pi}{4}\right) = \frac{1}{k}$, where k is a whole number. Find the value of k.

Solution:

$$\frac{1}{2}\int \cos^3 2x dx = \frac{1}{2}\int \cos^2 2x \cos 2x dx = \frac{1}{2}\int \left(1-\sin^2 2x\right)\cos 2x dx$$

Letting $u = \sin 2x$ gives $du = 2\cos 2x dx$. Substituting in the integrand yields,

$$\frac{1}{2}\int \left(1-\sin^2 2x\right)\cos 2x dx = \frac{1}{4}\int \left(1-u^2\right)du = \frac{1}{4}\left(u - \frac{u^3}{3}\right) + C = \frac{1}{4}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + C \,.$$

Hence, $I\left(x\right) = \frac{1}{4}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)$**.**

Substituting $\frac{\pi}{4}$ for x in the expression for $I(x)$ gives,

$$I\left(x\right) = \frac{1}{4}\left(\sin\frac{\pi}{2} - \frac{\sin^3\frac{\pi}{2}}{3}\right) = \frac{1}{4}\left(1-\frac{1}{3}\right) = \frac{1}{6}.$$

Therefore, the value of k is 6.

2. Question Number: 74
Desc: Integrate the cube of sine function

$12\int_{\pi/12}^{\pi/6} \sin^3 4x dx = \frac{k}{4}$, where k is a constant to be determined. What is the value of k?

Solution:

$$12\int_{\pi/12}^{\pi/6} \sin^3 4x dx = -3\int_{\pi/12}^{\pi/6}\left(1-\cos^2 4x\right)d\left(\cos 4x\right) = -3\left(\cos 4x - \frac{\cos^3 4x}{3}\right)\Bigg|_{\pi/12}^{\pi/6} =$$

$$-3\left(\cos\frac{2\pi}{3} - \frac{\cos^3\frac{2\pi}{3}}{3} - \cos\frac{\pi}{3} + \frac{\cos^3\frac{\pi}{3}}{3}\right) = -3\left(-\frac{1}{2}+\frac{1}{24}-\frac{1}{2}+\frac{1}{24}\right) = \frac{11}{4}$$

Therefore, $k = 11$.

3. Question Number: 75
Desc: Integrate the square of a cosine function

$\int \cos^2 12x \, dx = \frac{1}{m} x + \frac{1}{n} \sin 24x + C$, where m and n are constants to be determined while C is the constant of integration.
What is the value of $m \times n$?

Solution:, $\int \cos^2 12x \, dx = \int \frac{1+\cos 24x}{2} \, dx = \frac{1}{2}\left(x + \frac{\sin 24x}{24} \right) + C = \frac{x}{2} + \frac{\sin 24x}{48} + C$
Therefore, $m = 2$**,** $n = 48$ **and** $m \times n = 96$**.**

4. Question Number: 76
Desc: Integrate using a trigonometric substitution

Consider the integral $\int \frac{x^2}{\sqrt{100 - x^2}} dx$.
Which of the following is true?

Select all that apply and enter their labels in the same order as they appear (ascending order).
Enter the labels without any spaces or commas.

1. The substitution $x = \sin 10\theta$ is helpful in evaluating the integral.

2. The substitution $x = 10 \sin \theta$ is helpful in evaluating the integral.

3. Applying a proper substitution (not necessarily as in **1.** or **2.** above) reduces the integral to $\int 100 \tan^2 \theta \, d\theta$.

4. Applying a proper substitution (not necessarily as in **1.** or **2.** above) reduces the integral to $\int 100 \sin^2 \theta \, d\theta$.

5. $\int \frac{x^2}{\sqrt{100 - x^2}} dx = 50 \sin^{-1} \frac{x}{10} - \frac{x}{2} \sqrt{100 - x^2} + C$

Solution:
Using the substitution $x = 10 \sin \theta$**, we obtain**

$$\int \frac{1000 \sin^2 \theta \cos \theta}{\sqrt{100 - 100 \sin^2 \theta}} d\theta = \int \frac{1000 \sin^2 \theta \cos \theta}{10 \cos \theta} d\theta = 50 \int 2 \sin^2 \theta d\theta = 50 \int (1 - \cos 2\theta) d\theta =$$

$$50\left(\theta-\frac{\sin 2\theta}{2}\right)+C=50\left(\sin^{-1}\left(\frac{x}{10}\right)-\sin\theta\cos\theta\right)+C=50\left(\sin^{-1}\left(\frac{x}{10}\right)-\frac{x}{10}\sqrt{1-\left(\frac{x}{10}\right)^2}\right)+C=$$

$$50\sin^{-1}\left(\frac{x}{10}\right)-\frac{x}{2}\sqrt{100-x^2}+C$$

So the substitution $10\sin\theta$ is helpful in evaluating the integral, it reduces the integral to $\int 100\sin^2\theta\, d\theta$ and gives $\int\frac{x^2}{\sqrt{100-x^2}}dx=50\sin^{-1}\left(\frac{x}{10}\right)-\frac{x}{2}\sqrt{100-x^2}+C$, therefore, the answer is 245.

5. Question Number: 77
Desc: Integrate the product of the squares of sine and cosine

$128\int\cos^2 2x\sin^2 2x dx = mx - n\sin kx + C$, where m, n, and k are constants to be determined while C is the constant of integration.
What is the value of $m \times n \times k$?

Solution: $128\int\cos^2 2x\sin^2 2x dx=128\int\frac{\sin^2 4x}{4}dx=32\int\sin^2(4x)dx=32\int\frac{1-\cos(8x)}{2}dx=$

$2\int\left(1-\cos(8x)\right)d(8x)=2(8x)-2\sin 8x=16x-2\sin 8x+C$

Therefore, m = 16, n = 2, k = 8 and $m \times n \times k$ = 256.

Lesson topic: 18. Partial Fractions

1. Question Number: 78
Desc: Find the partial fractions of a given rational expression

If $\frac{3x^3-3x+2}{x(x-2)(x+1)}=3-\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}$, what is the value of ($A$ + B + C)? Enter the answer as a whole number.

Solution:

Multiply both sides of $\frac{3x^3-3x+2}{x(x-2)(x+1)}=3-\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+1}$ by $x(x-2)(x+1)$. This gives,

$3x^3-3x+2=3x(x-2)(x+1)-A(x-2)(x+1)+Bx(x+1)+Cx(x-2)$ **(i).**

The last equation is true for any value of x, even for x = 0, 2, or –1.
Substituting x = 0 in (i) yields, 2 = 2A or A = 1.

Substituting x = 2 in (i) yields, 20 = 6B or $B=\frac{10}{3}$.

Substituting x = –1 in (i) yields, 2 = 3C or $C=\frac{2}{3}$.

Therefore, the value of A + B + C is $1+\frac{10}{3}+\frac{2}{3}=5$.

2. Question Number: 79
Desc: Integrate using partial fractions

$\int\frac{2x+7}{x^2+10x+25}dx=a\ln|x+5|+\frac{b}{x+5}+C$, where a and b are constants to be determined while C is the constant of integration.
What is the value of $a\times b$?

Solution:,

$$\int\frac{2x+7}{x^2+10x+25}dx=\int\frac{2(x+5)-3}{(x+5)^2}dx=\int\left(\frac{2}{x+5}-\frac{3}{(x+5)^2}\right)dx=2\ln|x+5|+\frac{3}{x+5}+C$$

Hence, a = 2, b = 3 and $a\times b$ = 6.

Lesson topic: 19. Integration by Parts

1. Question: 80
Desc: Integrate x^n x e^x using integration by parts

If $\int\left(3x^2+2x+1\right)e^x dx = I(x)+C$, what is the value of $I(1)$ rounded to the nearest whole number?

Solution:
Let $u=3x^2+2x+1$ and $dv=e^x dx$. $du=\left(6x+2\right)dx$ and $v=e^x$.

$$\int\left(3x^2+2x+1\right)e^x dx=\left(3x^2+2x+1\right)e^x-\int\left(6x+2\right)e^x dx \quad \text{(i)}$$

Let $u=6x+2$ and $dv=e^x dx$. $du=6dx$ and $v=e^x$.

$$\int\left(6x+2\right)e^x dx=\left(6x+2\right)e^x-\int 6e^x dx=\left(6x+2\right)e^x-6e^x=\left(6x-4\right)e^x \quad \text{(ii)}$$

Substituting (ii) into (i) gives,

$$\int\left(3x^2+2x+1\right)e^x dx=\left(3x^2+2x+1\right)e^x-\int\left(6x+2\right)e^x dx=\left(3x^2+2x+1\right)e^x-\left(6x-4\right)e^x+C$$

$$=\left(3x^2-4x+5\right)e^x+C.$$

Hence, $I\left(x\right)=\left(3x^2-4x+5\right)e^x$.

Substituting 1 for x in the expression for $I(x)$ gives,

$$I\left(1\right)=\left(3\left(1\right)^2-4\left(1\right)+5\right)e^1=4e\approx 10.87.$$

Therefore, the value of $I(1)$, rounded to the nearest whole number, is 11.

2. Question: 81
Desc: Integrate x^n sin x using integration by parts

Using the rounded values cos 4 = –0.7 and sin 4 = – 0.8, evaluate the integral $\int_0^4 11x\sin x\, dx$, and enter your answer as a whole number.

Solution:
Let $u=11x$ and $dv=\sin x dx$. $du=11dx$ and $v=-\cos x$.

$$\int_0^4 11x\sin x dx=-11x\cos x\Big|_0^4+\int_0^4 11\cos x dx=-44\cos 4+11\sin x\Big|_0^4==11\left(\sin 4-4\cos 4\right).$$

Substituting the rounded values of cos 4 = –0.7 and sin 4 = – 0.8 in the last expression gives,

$$\int_0^4 11x\sin x\,dx = 11(-0.8 + 4\times 0.7) = 22.$$

Therefore, the value of the integral $\int_0^4 11x\sin x\,dx$ is 22.

Lesson topic: 20. Numerical integration

1. Question: 82
Desc: Use the trapezoidal formula to approximate a definite integral

Which of the following is closest to the estimated value of the integral $I = \int_0^1 \cos\left(x^2 + 1\right)dx$ obtained using the trapezoidal formula with n = 4?
Enter the label of the proper choice.

1. 0.2

2. 0.3

3. 0.4

4. 0.5

5. 0.6

6. 1

Solution:
***a* = 0, *b* = 1, *h* = 0.25, x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1.**

$$\boldsymbol{I \approx I_4 = \frac{h}{2}\left[f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + 2f\left(x_3\right) + f\left(x_4\right)\right]}$$

$$\mathbf{= \frac{0.25}{2}\left[f\left(0\right) + 2f\left(0.25\right) + 2f\left(0.5\right) + 2f\left(0.75\right) + f\left(1\right)\right]}$$

$$\mathbf{= \frac{0.25}{2}\left[0.5403 + 0.9734 + 0.6306 + 0.0166 - 0.1461\right] = 0.2181}$$

(All calculations are rounded to four decimal places).
Therefore, the estimated value 0.2181 is closest to 0.2 and the correct option is 1.

2. Question: 83
Desc: Find a bound on the error when using the trapezoidal formula

Which of the following is closest to the error bound on the approximation of the integral $I = \int_0^1 \left(6x^3 - 3x^2 + 10\right)dx$ obtained using the trapezoidal formula with n = 9.

Enter the label of the proper choice.

1. 0.02

2. 0.03

3. 0.05

4. 0.06

5. 0.07

Solution:
$f(x) = 6x^3 - 3x^2 + 10$, $f'(x) = 18x^2 - 6x$ **and** $f''(x) = 36x - 6$. **The maximum value *M* of** $|f''(x)|$ **on [0, 1] is 30.**

Using *n* = 9, $h = \frac{1-0}{9} = \frac{1}{9}$ **yields,** $E_T \le \left|\frac{(b-a)h^2}{12} M\right| = \left|\frac{\cancel{30}^{5}}{\cancel{81}_{27} \times \cancel{12}_{6}}\right| = \frac{5}{162} \approx 0.031$.

The option 2 is closest to the error bound on the approximation of the integral.
Therefore, the answer is 2.

Lesson topic: 21. Lines in space

1. Question Number: 84
Desc: Find the vector equation of a line in space

Consider the vector equation of the line *l* that passes through *A*(1, 2, 1) and parallel to **u** = 3**i** + **j** + 5**k**.

Suppose that the point (4, 3, 6) corresponds to $t = 1$ and the point (7, 4, 11) corresponds to $t = 2$ then what is the *x*-coordinate of the point on *l* that corresponds to $t = 10$?

[Hint: Find the vector equation of *l* as you normally do and substitute 10 for *t*.
The points corresponding to $t = 1$ and $t = 2$ are given to ensure a unique answer.]

Solution:
The equation of the line that passes through *A*(1, 2, 1) and with a directional vector

v = 3i + j + 5k is $\begin{cases} x = 1 + 3t \\ y = 2 + t \\ z = 1 + 5t \end{cases}, t \in \mathbf{R}$.

The *x*-coordinate of the point on *l* that corresponds to *t* = 10 is 1 + 3 × 10 = 31.

2. Question: 85
Desc: Find and use the parametric equations of a line given a point and directions

The parametric equations of the line passing through the point *P*(7, 0, –4) and parallel to the vector ***u*** = ***i*** + 4***k*** are $x = p + qt$, $y = 0$, and $z = r + st$, where *p*, *q*, *r*, and *s* are constants. If the point *Q* (*a*, 0, 0) lies on this line, find the value of *a* using the parametric equations after determining the corresponding constants.

Solution:

Substituting the coordinates of *P* and the components of *u* into the equation $\begin{cases} x = x_0 + lt \\ y = y_0 + mt \\ z = z_0 + nt \end{cases}$

gives, $\begin{cases} x = 7 + t \\ y = 0 + 0 \times t \\ z = -4 + 4t \end{cases}$ **or** $\begin{cases} x = 7 + t \\ y = 0 \\ z = -4 + 4t \end{cases}$ **(i).**

Hence, *p* = 7, *q* = 1, *r* =–4, and *s* = 4.

Substituting the coordinates of Q into (i) yields, $\begin{cases} a = 7 + t \\ 0 = 0 \\ 0 = -4 + 4t \end{cases}$.

Finding t = 1 from the last equation and substituting it in the first one gives, a = 8.
Therefore, the value of a = 8.

3. Question: 86
Desc: Find the cartesian equation of a line in space

Consider the line (l) that passes through $A(0, 1, 2)$ and $B(2, 2, 5)$. If C is a point on l whose x-coordinate is 20, then what whole number must be equal to its y-coordinate?
Find the Cartesian equation of l and substitute 20 for x and solve for y. Simplify your answer.

Solution:
The equation of the line that passes through $A(0, 1, 2)$ and $B(2, 2, 5)$ is
$$\frac{x-0}{2-0} = \frac{y-1}{2-1} = \frac{z-2}{5-2}, \text{ or } \frac{x}{2} = \frac{y-1}{1} = \frac{z-2}{3}.$$
If x = 20 then $\frac{20}{2} = \frac{y-1}{1} \Leftrightarrow y = 11.$

4. Question: 87
Desc: Find the distance between a point and a line in space

The distance from $A(1, 1, 0)$ to (l): $\begin{cases} x = 1 + t \\ y = 3 - t \\ z = 1 + 2t \end{cases}$, $t \in \mathbf{R}$, is $\sqrt{a}$. What is the value of a?

Solution:
Consider the point $B(2, 2, 3)$ on (l).
$$\overrightarrow{AB} = \mathbf{i} + \mathbf{j} + 3\mathbf{k},\ v = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$$
$$\overrightarrow{AB} \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$
$$\left|\overrightarrow{AB} \times v\right| = \sqrt{5^2 + 1^2 + (-2)^2} = \sqrt{30},\ |v| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$$
The distance from $A(1, 1, 0)$ to (l) equals $d = \frac{\left|\overrightarrow{AB} \times v\right|}{|v|} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$

Therefore, a = 5.

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