Lesya Ukrainka Volyn National University

Department of General Mathematics and Computer Science Teaching Methods

Maria Khomyak

Inna Mykytyuk

MATHEMATICS: Integration Techniques: methodological workshop on problem solving

Lutsk 2024

UDK 517.3(076)
Kh 76

Recommended for publication by the Scientific and Methodological Council of Lesya Ukrainka Volyn National University (protocol No 1, 25.09.2024)

Reviewer:

V. V. Pasternak, Ph.D. in Technical sciences, Assoc. Department of General Mathematics and Computer Science Teaching Methods of Lesya Ukrainka Volyn National University

Khomyak M., Mykytyuk I.

Kh 76 MATHEMATICS: Integration Techniques: methodological workshop on problem solving / Maria Khomyak, Inna Mykytyuk. Lutsk : Lesya Ukrainka VNU, 2024. 90 p.

Annotation: The methodological workshop is given curriculum for the discipline "Mathematics" or "High Mathematics" of the professional training of the bachelor of non-mathematical specialties.

The methodological workshop contains summary of lecture material and typical examples along with their solutions on some specific topics on Integration Techniques. The methodological workshop offers also exercises for students for solving by themselves. It could be useful in preparation for practical classes as well as the independent work and also for quizzes and the final test or exam.

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Lecture summary

To *integrate expressions of the form* $\cos^n x \sin^m x$ when either n or m (or both) is odd choose the expression with the odd exponent, say $\sin^m x$, and express it as

$$\sin^{m-1} x \ \sin x = \left(\sin^2 x\right)^{\frac{m-1}{2}} \sin x = \left(1-\cos^2 x\right)^{\frac{m-1}{2}} \sin x \ .$$

The integrand $\cos^n x \sin^m x$ becomes $\cos^n x \left(1-\cos^2 x\right)^{\frac{m-1}{2}} \sin x = \left[P(\cos x)\right](\sin x)$ where $P(\cos x)$ is a polynomial in $\cos x$. Finally, use the substitution $u = \cos x$.

To *integrate expressions of the form* $\cos^n x \sin^m x$ when both n and m are even, reduce the order of both factors by writing $\sin^m x = \left(\sin^2 x\right)^{m/2}$ and

$\cos^n x = \left(\cos^2 x\right)^{n/2}$ and using the identities $\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{1+\cos 2x}{2}$.

The formulas for the integrals of simple trigonometric expressions are:

- $\int \sin u \ du = -\cos u + C$
- $\int \cos u \ du = \sin u + C$
- $\int \sec^2 u \ du = \tan u + C$
- $\int \csc^2 u \ du = -\cot u + C$
- $\int \sec u \tan u \ du = \sec u + C$
- $\int \csc u \cot u \ du = -\csc u + C$
- $\int \tan u \ du = \ln|\sec u| + C$
- $\int \cot u \ du = -\ln|\csc u| + C$
- $\int \sec u \ du = \ln|\sec u + \tan u| + C$
- $\int \csc u \ du = -\ln|\csc u + \cot u| + C$

Using the above formulas, the principle of substitution, and manipulating trigonometric identities the integral of many algebraic expressions may be evaluated.

Expressions involving $\sqrt{a^2 - x^2}$

Let $\theta = \sin^{-1}\dfrac{x}{a}$ then $x = a \sin\theta$ and $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\left(1-\sin^2\theta\right)} =$

$\sqrt{a^2\cos^2\theta} = a\cos\theta$ as a and $\cos\theta$ are both positive.

Expressions involving $\sqrt{a^2 + x^2}$

Let $\theta = \tan^{-1}\dfrac{x}{a}$ then $x = a \tan\theta$ and $\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2\left(1+\tan^2\theta\right)} =$

$\sqrt{a^2\sec^2\theta} = a\sec\theta$ as a and $\sec\theta$ are both positive.

Expressions involving $\sqrt{x^2-a^2}$

Let $\theta=\sec^{-1}\left|\dfrac{x}{a}\right|$ then $|x|=a\sec\theta$ and $\sqrt{x^2-a^2}=\sqrt{a^2\sec^2\theta^2-a^2}=$ $\sqrt{a^2\left(\sec^2\theta-1\right)}=\sqrt{a^2\tan^2\theta}=a\tan\theta$ as a and $\tan\theta$ are both positive.

The *method of partial fractions* is applied to proper rational expressions (degree of numerator is less than the degree of the denominator). To apply partial fractions, improper rational expressions must be first reduced to a quotient plus a proper fraction. The method of partial fractions is then applied to the proper fraction.

To expand $\dfrac{x^3+2x+1}{(x+2)(x-1)}$ using partial fractions $\dfrac{x^3+2x+1}{(x+2)(x-1)}$ is expressed as $x-1+\dfrac{5x-1}{(x+2)(x-1)}$ and then the method of partial fractions is applied to $\dfrac{5x-1}{(x+2)(x-1)}$.

Suppose that $P(x)$ is a polynomial of degree $k-1$ or less and none of its roots is equal to r_i, $i=1,\dots,k$. $\dfrac{P(x)}{(x-r_1)(x-r_2)\dots(x-r_k)}$ can be expressed as $\dfrac{A_1}{x-r_1}+\dfrac{A_2}{x-r_2}+\dots+\dfrac{A_k}{x-r_k}$

where $A_i=\left.\dfrac{P(x)}{(x-r_1)(x-r_2)\dots\cancel{(x-r_i)}\dots(x-r_k)}\right]_{x=r_i}$.

For example, if $\dfrac{4x-16}{x(x-1)(x+2)}=\dfrac{A_1}{x}+\dfrac{A_2}{x-1}+\dfrac{A_3}{x+2}$ then

$A_1=\left.\dfrac{4x-16}{\cancel{x}(x-1)(x+2)}\right]_{x=0}=\dfrac{-16}{-2}=8$, $A_2=\left.\dfrac{4x-16}{x\cancel{(x-1)}(x+2)}\right]_{x=1}=\dfrac{-12}{3}=-4$, and

$A_3=\left.\dfrac{4x-16}{x(x-1)\cancel{(x+2)}}\right]_{x=-2}=\dfrac{-24}{6}=-4$.

If multiple roots are present as in $\dfrac{P(x)}{(x-r_1)^{n_1}(x-r_2)^{n_2}\dots(x-r_k)^{n_k}}$ the expansion becomes

$$\frac{A_{11}}{(x-r_1)^{n_1}}+\frac{A_{12}}{(x-r_1)^{n_1-1}}+\dots+\frac{A_{1n_1}}{x-r_1}+\frac{A_{21}}{(x-r_2)^{n_2}}+\frac{A_{22}}{(x-r_2)^{n_2-1}}+\dots+\frac{A_{2n_2}}{x-r_2}+\dots$$

$$A_{11}=\left.\frac{P(x)}{\cancel{(x-r_1)^{n_1}}(x-r_2)^{n_2}\dots(x-r_k)^{n_k}}\right]_{x=r_1},$$

$$A_{12}=\left.\left(\frac{P(x)}{\cancel{(x-r_1)^{n_1}}(x-r_2)^{n_2}\dots(x-r_k)^{n_k}}\right)'\right]_{x=r_1},$$

$$A_{13}=\left.\frac{1}{2!}\left(\frac{P(x)}{\cancel{(x-r_1)^{n_1}}(x-r_2)^{n_2}\dots(x-r_k)^{n_k}}\right)''\right]_{x=r_1},\ \dots$$

The expansion of expansions such as $\frac{P(x)}{(ax^2+bx+c)(x-r)}$ where $ax^2 + bx + c$ has no real roots has the form $\frac{Ax+B}{ax^2+bx+c}+\frac{C}{x-r}$.

The method of partial fractions is a good tool in *finding the integrals of rational expressions* whose integral are not directly recognized.

Using $u = f(x)$ and $dv = h(x)\,dx$ in $\int f(x)h(x)\,dx$ transforms the integral to $\int u\,dv$ which is equal to $uv - \int v\,du$. If finding v and the integral $\int v\,du$ are both easy then the integral $\int f(x)h(x)\,dx$ can be evaluated. This technique of finding anti-derivatives is called *Integration by parts*.

When $p(x)$ is a polynomial and $h(x)$ is a function whose anti-derivatives can be found easily then applying integration by parts to $\int p(x)h(x)\,dx$ can be performed *by tabulating* in one column $p(x)$ and its derivatives and on the other $h(x)$ and its anti-derivatives:

$(+)\ p$	h
$(-)\ p'$	$\int h\,dx \equiv H$
$(+)\ p''$	$\int H\,dx$
$\vdots$	

The *trapezoidal rule* is a method for finding approximate value for the integral:

$$I = \int_a^b f(x)\,dx \approx I_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

where $h = \frac{b-a}{n}$, $x_k = a + kh$, and n is a positive integer.

For a twice differentiable function f, the difference between the exact value of $I = \int_a^b f(x)\,dx$ and the approximation, I_n, obtained using the trapezoidal rule is given by E_T which satisfies $E_T \le \left|\frac{(b-a)h^2}{12}M\right|$ where $|f''(x)| \le M$ on $[a, b]$.

Another method for finding an approximate value for the integral $I = \int_a^b f(x)\,dx$ is *Simpson's rule*:

$$I \approx I_n \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where $h = \frac{b-a}{n}$, $x_k = a + kh$, and n is an even positive integer.

For a four times differentiable function f, the difference between the exact value of $I = \int_a^b f(x)\,dx$ and the approximation I_n obtained using the Simpson's rule is given by E_S which satisfies $E_S \le \left|\frac{(b-a)h^4}{180}M\right|$ where $|f^{(4)}(x)| \le M$ on $[a, b]$.

1. Trigonometric Expressions

1.1. The integral of $\sin^m x \cos^n x$

1. $\int \sin 2x \, dx = -\frac{\cos 2x}{2} + C.$

2. $\int_0^{\pi/2} \cos \frac{1}{2} y \, dy = 2 \sin \frac{y}{2}\Big|_0^{\pi/2} = 2(\sin \frac{\pi}{4} - \sin 0) = 2(\frac{\sqrt{2}}{2} - 0) = \sqrt{2}.$

3. $\int e^{2z} \cos\left(e^{2z}\right) dz = \frac{1}{2} \int \cos u \, du$ $\qquad (u = e^{2z})$

$$= \frac{1}{2} \sin u + C = \frac{1}{2} \sin e^{2z} + C.$$

4. $\int_{\pi^2/36}^{\pi^2/4} \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx = 2 \int_{\pi/6}^{\pi/2} \cos u \, du$ $\qquad (u = \sqrt{x})$

$$= 2 \sin u \Big|_{\pi/6}^{\pi/2} = 2(\sin \frac{\pi}{2} - \sin \frac{\pi}{6}) = 2(1 - \frac{1}{2}) = 1.$$

5. $\int \frac{\sin(\ln w)}{w} \, dw = \int \sin u \, du$ $\qquad (u = \ln w)$

$$= -\cos u + C = -\cos(\ln w) + C.$$

6. $\int t \cos\left(t^2 + 1\right) dt = \frac{1}{2} \int \cos u \, du$ $\qquad (u = t^2 + 1)$

$$= \frac{1}{2} \sin u + C = \frac{1}{2} \sin(t^2 + 1) + C.$$

7. $\int_0^{\pi/2} \sin x \cos^5 x \, dx = -\int_1^0 u^5 \, du$ $\qquad (u = \cos x)$

$$= \frac{u^6}{6}\Big|_0^1$$

$$= \frac{1}{6}(1 - 0) = \frac{1}{6}.$$

8. $\int \sin^5 p \cos p \, dp = \int u^5 \, du$ $\qquad (u = \sin p)$

$$= \frac{u^6}{6} + C = \frac{\sin^6 p}{6} + C$$

9. $\int \sin^2 4t \, dt = \int \frac{1 - \cos 8t}{2} \, dt$

$$= \frac{1}{2} \int dt - \frac{1}{2} \int \cos 8t \, dt$$

$$= \frac{1}{2}t - \frac{1}{2}\frac{\sin 8t}{8} + C$$

$$= \frac{1}{2}t - \frac{1}{16}\sin 8t + C.$$

10. $\int_0^{\pi} \cos^2 4q\, dq = \int_0^{\pi} \frac{1+\cos 8q}{2}\, dq$

$$= \frac{1}{2}\int_0^{\pi} dq + \frac{1}{2}\int_0^{\pi} \cos 8q\, dq$$

$$= \frac{1}{2}q\Big|_0^{\pi} + \frac{1}{2}\cdot\frac{\sin 8q}{8}\Big|_0^{\pi}$$

$$= \frac{1}{2}(\pi - 0) + \frac{1}{16}(\sin 8\pi - \sin 0) = \frac{\pi}{2}.$$

11. $\int \sin^3 2x\, dx = \int \sin^2 2x \sin 2x\, dx$

$$= \int \frac{1-\cos 4x}{2}\sin 2x\, dx$$

$$= \frac{1}{2}\int \sin 2x dx - \frac{1}{2}\int \cos 4x \sin 2x\, dx$$

$$= \frac{1}{2}\int \sin 2x dx - \frac{1}{4}\int (\sin 6x + \sin 2x)\, dx$$

$$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) - \frac{1}{4}\left(-\frac{\cos 6x}{6}\right) - \frac{1}{4}\left(-\frac{\cos 2x}{2}\right) + C$$

$$= \frac{\cos 6x}{24} - \frac{\cos 2x}{8} + C.$$

12. $\int \cos^2 2x \sin^4 2x\, dx = \int \cos^2 2x \sin^2 2x\, \sin^2 2x dx$

$$= \frac{1}{4}\int (2\cos 2x \sin 2x)^2\, \sin^2 2x dx$$

$$= \frac{1}{4}\int \sin^2 4x\, \sin^2 2x dx$$

$$= \frac{1}{4}\int \sin^2 4x \left(\frac{1-\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int \sin^2 4x \left(\frac{1-\cos 4x}{2}\right) dx$$

$$= \frac{1}{8}\int \sin^2 4x \left(1-\cos 4x\right) dx$$

$$= \frac{1}{8}\int \sin^2 4x\, dx - \frac{1}{8}\int \sin^2 4x \cos 4x dx$$

$$= \frac{1}{8}\int \frac{1-\cos 8x}{2}\, dx - \frac{1}{8}\int \frac{u^2}{4}\, du \qquad (u = \sin 4x)$$

$$= \frac{1}{16}x - \frac{1}{16}\frac{\sin 8x}{8} - \frac{1}{32}\cdot\frac{u^3}{3} + C$$

$$= \frac{1}{16}x - \frac{\sin 8x}{128} - \frac{\sin^3 4x}{96} + C$$

13. $\int \cos^4 2y\, dy = \frac{1}{4}\int (2\cos^2 2y)^2\, dy$

$$= \frac{1}{4}\int (1+\cos 4y)^2\, dy$$

$$= \frac{1}{4}\int (1+2\cos 4y+\cos^2 4y)\, dy$$

$$= \frac{1}{4}\int (1+2\cos 4y+\frac{1+\cos 8y}{2})\, dy$$

$$= \frac{1}{4}(y+2\frac{\sin 4y}{4}+\frac{1}{2}y+\frac{1}{2}\frac{\sin 8y}{8}) + C$$

$$= \frac{3}{8}y+\frac{1}{8}\sin 4y+\frac{1}{64}\sin 8y + C.$$

14. $\int_0^1 \sin^4 \pi x\, dx = \frac{1}{4}\int_0^1 (2\sin^2 \pi x)^2\, dx$

$$= \frac{1}{4}\int_0^1 (1-\cos 2\pi x)^2\, dx$$

$$= \frac{1}{4}\int_0^1 (1-2\cos 2\pi x+\cos^2 2\pi x)\, dx$$

$$= \frac{1}{4}\int_0^1 (1-2\cos 2\pi x+\frac{1+\cos 4\pi x}{2})\, dx$$

$$= \frac{1}{4}\int_0^1 (\frac{3}{2}-2\cos 2\pi x+\frac{1}{2}\cos 4\pi x)\, dx$$

$$= \left(\frac{3}{8}x-\frac{1}{4\pi}\sin 2\pi x+\frac{1}{32\pi}\cos 4\pi x\right)\Big|_0^1$$

$$= \frac{3}{8}-\frac{1}{4\pi}\sin 2\pi+\frac{1}{32\pi}\cos 4\pi-\frac{1}{32\pi}$$

$$= \frac{3}{8} - 0 + 0 = \frac{3}{8}.$$

15. $\int \cos^3 3\theta \sin^2 3\theta \, d\theta = \int \cos^2 3\theta \sin^2 3\theta \cos 3\theta d\theta$

$$= \int (1 - \sin^2 3\theta) \sin^2 3\theta \cos 3\theta d\theta$$

$$= \int (1 - \sin^2 3\theta) \sin^2 3\theta \cos 3\theta d\theta$$

$$= \frac{1}{3}\int (1 - u^2) u^2 \, du \qquad (u = \sin 3\theta)$$

$$= \frac{1}{3}\int (u^2 - u^4) du$$

$$= \frac{1}{3}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

$$= \frac{u^3}{9} - \frac{u^5}{15} + C$$

$$= \frac{\sin^3 3\theta}{9} - \frac{\sin^5 3\theta}{15} + C.$$

16. $\int \cos^6 2x \, dx = \frac{1}{8}\int (2\cos^2 2x)^3 \, dx$

$$= \frac{1}{8}\int (1 + \cos 4x)^3 \, dx$$

$$= \frac{1}{8}\int (1 + 3\cos 4x + 3\cos^2 4x + \cos^3 4x) \, dx$$

$$= \frac{1}{8}\int (1 + 3\cos 4x + \frac{3}{2}(1 + \cos 8x) + (1 - \sin^2 4x)\cos 4x) \, dx$$

$$= \frac{1}{8}\int (1 + 3\cos 4x + \frac{3}{2}(1 + \cos 8x)) \, dx + \frac{1}{8}\int (1 - u^2) \frac{du}{4} \qquad (u = \sin 4x)$$

$$= \frac{1}{8}\left(x + \frac{3\sin 4x}{4} + \frac{3}{2}(x + \frac{\sin 8x}{8})\right) + \frac{1}{32}\left(u - \frac{u^3}{3}\right) + C$$

$$= \frac{5}{16}x + \frac{\sin 4x}{8} + \frac{3\sin 8x}{128} - \frac{\sin^3 4x}{96} + C$$

17. $\int \cos^3 3y \, dy = \int \cos^2 3y \cos 3y \, dy$

$$= \int (1 - \sin^2 3y) \cos 3y \, dy \qquad (u = \cos 3y)$$

$$= \frac{1}{3}\int (1 - u^2) \, du$$

$$= \frac{1}{3}(u - \frac{u^3}{3}) + C$$

$$= \frac{u}{3} - \frac{u^3}{9} + C$$

$$= \frac{\cos 3y}{3} - \frac{\cos^3 3y}{9} + C \ .$$

18. $\int \cos^3 2s \sin^3 2s \ ds = \frac{1}{8}\int (2\cos 2s \sin 2s)^3 \ ds$

$$= \frac{1}{8}\int \sin^3 4s \ ds$$

$$= \frac{1}{8}\int \sin^2 4s \ \sin 4s \, ds$$

$$= \frac{1}{8}\int (1-\cos^2 4s)\sin 4s \, ds \qquad (u = \cos 4s)$$

$$= -\frac{1}{32}\int (1-u^2)du$$

$$= -\frac{1}{32}(u - \frac{u^3}{3}) + C$$

$$= \frac{\cos^3 4s}{96} - \frac{\cos 4s}{32} + C.$$

19. $\int \sin^6 4s \ ds = \frac{1}{8}\int (2\sin^2 4s)^3 \ ds$

$$= \frac{1}{8}\int (1-\cos 8s)^3 \ ds$$

$$= \frac{1}{8}\int (1 - 3\cos 8s + 3\cos^2 8s - \cos^3 8s) \ ds$$

$$= \frac{1}{8}\int (1 - 3\cos 8s + \frac{3(1+\cos 16s)}{2} - (1-\sin^2 8s)\cos 8s) \ ds$$

$$= \frac{1}{8}\int (1 - 3\cos 8s + \frac{3(1+\cos 16s)}{2})ds - \frac{1}{8}\int (1-\sin^2 8s)\cos 8s \ ds \qquad (u = \sin 8s)$$

$$= \frac{1}{8}\int (1 - 3\cos 8s + \frac{3(1+\cos 16s)}{2})ds - \frac{1}{8}\int \frac{(1-u^2)}{8} \ du$$

$$= \frac{1}{8}\left(s - \frac{3\sin 8s}{8} + \frac{3s}{2} + \frac{3\sin 16s}{32} - \frac{1}{8}(u - \frac{u^3}{3}) \right) + C$$

$$= \frac{5s}{16} - \frac{3\sin 8s}{64} + \frac{3\sin 16s}{256} - \frac{\sin 8s}{64} + \frac{\sin^3 8s}{192} + C$$

$$= \frac{5s}{16} - \frac{\sin 8s}{16} + \frac{3\sin 16s}{256} + \frac{\sin^3 8s}{192} + C$$

20. $\int \cos^5 2r\ dr = \int \cos^4 2r \cos 2r\, dr$

$$= \int (1-\sin^2 2r)^2 \cos 2r\, dr \qquad (u = \sin 2r)$$

$$= \frac{1}{2}\int (1-u^2)^2\ du$$

$$= \frac{1}{2}\int (1-2u^2+u^4)\ du$$

$$= \frac{1}{2}(u - \frac{2u^3}{3} + \frac{u^5}{5}) + C$$

$$= \frac{\sin 2r}{2} - \frac{2\sin^3 2r}{3} + \frac{\sin^5 2r}{10} + C.$$

21. $\int \cos^5 2w \sin^4 2w\ dw = \int \cos^4 2w \sin^4 2w \cos 2w dw$

$$= \int (1-\sin^2 2w)^2 \sin^4 2w \cos 2w dw \qquad (u = \sin 2w)$$

$$= \frac{1}{2}\int (1-u^2)^2 u^4\ du$$

$$= \frac{1}{2}\int (1-2u^2+u^4)u^4\ du$$

$$= \frac{1}{2}\int (u^4 - 2u^6 + u^8)\ du$$

$$= \frac{1}{2}(\frac{u^5}{5} - 2\frac{u^7}{7} + \frac{u^9}{9}) + C$$

$$= \frac{\sin^5 2w}{10} - \frac{\sin^7 2w}{7} + \frac{\sin^9 2w}{18} + C$$

22. $\int \frac{\cos^2(\ln r)}{r}\ dr = \int \cos^2 u\ du \qquad (u = \ln r)$

$$= \int \frac{(1+\cos 2u)}{2}\ du$$

$$= \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$

$$= \frac{1}{2}\ln r + \frac{1}{4}\sin(2\ln r) + C.$$

23. $\int e^x \sin^2(e^x)\ dx = \int \sin^2 u\ du \qquad (u = e^x)$

$$= \int \frac{(1-\cos 2u)}{2}\, du$$

$$= \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$

$$= \frac{1}{2}e^x - \frac{1}{4}\sin 2e^x + C$$

24. $\int \cos^3 2x \sin^5 2x\, dx = \int \cos^2 2x \sin^5 2x \cos 2x dx$

$$= \int (1-\sin^2 2x)\sin^5 2x \cos 2x dx \qquad (u = \sin 2x)$$

$$= \frac{1}{2}\int (1-u^2)u^5 du$$

$$= \frac{1}{2}\int (u^5 - u^7) du$$

$$= \frac{1}{2}\left(\frac{u^6}{6} - \frac{u^8}{8}\right) + C$$

$$= \frac{\sin^6 2x}{12} - \frac{\sin^8 2x}{16} + C.$$

25. $\int \sin 3x \cos 5x\, dx = \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx$

$$= \frac{1}{2}\left(-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right) + C$$

$$= \frac{\cos 2x}{4} - \frac{\cos 8x}{16} + C.$$

26. $\int \cos 2x \cos 4x\, dx = \frac{1}{2}\int (\cos 6x + \cos 2x)\, dx$

$$= \frac{1}{2}\left(\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right) + C$$

$$= \frac{\sin 6x}{12} + \frac{\sin 2x}{4} + C.$$

27. $\int \sin 3x \sin 2x\, dx = \frac{1}{2}\int (\cos 2x - \cos 6x)\, dx$

$$= \frac{1}{2}\left(\frac{\sin 2x}{2} - \frac{\sin 6x}{6}\right) + C$$

$$= \frac{\sin 2x}{4} - \frac{\sin 6x}{12} + C.$$

28. a) If $|a| \neq |b|$, $\int \sin ax \cos bx\, dx = \frac{1}{2}\int (\sin(a+b)x - \sin(a-b)x)\, dx$

$$= \frac{1}{2}\left(-\frac{\cos(a+b)x}{a+b} + \frac{\cos(a-b)x}{a-b}\right) + C$$

$$= \frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} + C.$$

b) $|a| \neq |b|$, $\int \sin ax \sin bx\, dx = \frac{1}{2}\int (\cos(a-b)x - \cos(a+b)x)\, dx$

$$= \frac{1}{2}\left(\frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b}\right) + C$$

$$= \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C.$$

c) $|a| \neq |b|$, $\int \cos ax \cos bx\, dx = \frac{1}{2}\int (\cos(a+b)x + \cos(a-b)x)\, dx$

$$= \frac{1}{2}\left(\frac{\sin(a+b)x}{a+b} + \frac{\sin(a-b)x}{a-b}\right) + C$$

$$= \frac{\sin(a+b)x}{2(a+b)} + \frac{\sin(a-b)x}{2(a-b)} + C.$$

***29** Let $\tan x = t$ then $x = \tan^{-1} t$ and $dx = \frac{1}{1+t^2}$.

$$\int_0^{\pi/2} \frac{1}{1+\tan^3 x}\, dx = \int_0^{\infty} \frac{1}{1+t^3} \cdot \frac{1}{1+t^2}\, dt.$$

$$\frac{1}{1+t^3} \cdot \frac{1}{1+t^2} = \frac{1}{(1+t)\left(1+t^2\right)\left(t^2-t+1\right)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2} + \frac{Dt+E}{t^2-t+1}.$$

$$\begin{cases} A+B+D=0 \\ -A+C+D+E=0 \\ 2A+D+E=0 \\ -A+B+D+E=0 \\ A+C+E=1 \end{cases} \Rightarrow \begin{cases} A=1/6 \\ B=1/2 \\ C=1/2 \\ D=-2/3 \\ E=1/3 \end{cases}.$$

Therefore, $\frac{1}{(1+t)\left(1+t^2\right)\left(t^2-t+1\right)} = \frac{1}{6(1+t)} + \frac{t+1}{2(1+t^2)} + \frac{-2t+1}{3(t^2-t+1)}.$

$$\int_0^{\infty} \frac{1}{1+t^3}\cdot\frac{1}{1+t^2}\,dt = \int_0^{\infty} \frac{1}{6(1+t)}\,dt + \int_0^{\infty} \frac{t+1}{2(1+t^2)}\,dt + \int_0^{\infty} \frac{-2t+1}{3(t^2-t+1)}\,dt = \frac{1}{6}\ln|1+t|\Big|_0^{\infty} +$$

$$\frac{1}{4}\ln|1+t^2|\Big|_0^{\infty} + \frac{1}{2}\tan^{-1}t\Big|_0^{\infty} - \frac{1}{3}\ln|t^2-t+1|\Big|_0^{\infty} = \ln\left(\frac{|1+t|^{1/2}\,(t^2+1)^{1/4}}{(1+t^3)^{1/3}}\right)\Bigg|_0^{\infty} + \frac{1}{2}\tan^{-1}t\Big|_0^{\infty} =$$

$$+\infty - 0 + \frac{1}{2}\cdot\frac{\pi}{2} - 0 = +\infty.$$

***30.** Since $g(x)$ is continuous on the closed interval $[0, \pi]$, exists c, $0 < c < \pi$, such that

$\int_0^{\pi} g(x)\,dx = g(c)\int_0^{\pi} dx = g(c)(\pi - 0) = \pi g(c)$ and $\pi g(c) = 0$.

Therefore, $g(c) = 0$ and c is a zero of $g(x)$.

Suppose that $g(x)$ has only one zero in the open interval $(0, \pi)$ and it is c.

Assume $g(x) > 0$ in $[0, c)$ and $g(x) < 0$ in $(c, \pi]$, and $c < \frac{\pi}{2}$. Then $\int_0^{c} g(x)\,dx = -\int_c^{\pi} g(x)\,dx$.

Moreover, $\int_0^{\pi/2} g(x)\cos x\,dx = \int_0^{c} g(x)\cos x\,dx + \int_c^{\pi/2} g(x)\cos x\,dx$ and $\cos x > \cos c$ as $0 < x < c$

implies $\int_0^{c} g(x)\cos x\,dx > \cos c\cdot\int_0^{c} g(x)\,dx$.

$$\left|\int_c^{\pi/2} g(x)\cos x\,dx\right| \le \cos c\cdot\left|\int_c^{\pi/2} g(x)\,dx\right| < \cos c\cdot\left|\int_c^{\pi} g(x)\,dx\right| = -\cos c\cdot\int_c^{\pi} g(x)\,dx.$$

Thus $\int_0^{c} g(x)\cos x\,dx > \cos c\cdot\int_0^{c} g(x)\,dx$ and $\int_c^{\pi} g(x)\cos x\,dx > \cos c\cdot\int_c^{\pi} g(x)\,dx$.

So $\int_0^{\pi} g(x)\cos x\,dx > \cos c\cdot\int_0^{\pi} g(x)\,dx$ which is impossible as both integrals equal to zero.

Therefore, $g(x)$ has at least two zeros in the open interval $(0, \pi)$.

1.2. Trigonometric substitutions

1. $\int \sec^2 5x\; dx \;=\; \int \frac{1}{\cos^2 5x}\, dx \;=\; \frac{1}{5}\tan 5x + C.$

2. $\int \csc 3r\; dr \;=\; \int \frac{1}{\sin 3r}\, dr \;=\; \frac{1}{2}\int \frac{1}{\sin\frac{3r}{2}\cos\frac{3r}{2}}\, dr$

$$= \frac{1}{2}\int \frac{1}{\tan\frac{3r}{2}\cos^2\frac{3r}{2}}\, dr \;=$$

$$= \frac{1}{3}\int \frac{1}{u}\, du \qquad\qquad (u = \tan\frac{3r}{2})$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln\left|\tan\frac{3r}{2}\right| + C$$

3. $\int \cot 10w\; dw \;=\; \int \frac{\cos 10w}{\sin 10w}\, dw$

$$= \frac{1}{10}\int \frac{1}{u}\, du \qquad\qquad (u = \sin 10w)$$

$$= \frac{1}{10}\ln|u| + C$$

$$= \frac{1}{10}\ln|\sin 10w| + C$$

4. $\int \sin 3x \tan 3x\; dx \;=\; \int \frac{\sin^2 3x}{\cos 3x}\, dx$

$$= \int \frac{1-\cos^2 3x}{\cos 3x}\, dx$$

$$= \int \left(\frac{1}{\cos 3x} - \cos 3x\right) dx$$

$$= \frac{1}{3}\ln|\sec 3x + \tan 3x| - \frac{1}{3}\sin 3x + C.$$

5. $\int_0^{\pi} \csc\frac{1}{4}y \cot\frac{1}{4}y\; dy \;=\; -4\csc\frac{y}{4}\Big|_0^{\pi} = -4\left(\frac{2}{\sqrt{2}} - \frac{1}{0}\right) = +\infty.$

6. $\displaystyle\int_{\pi/12}^{\pi/6} \csc^2 2x\, dx = -\frac{1}{2}\cot 2x\Big|_{\pi/12}^{\pi/6}$

$$= -\frac{1}{2}(\cot\frac{\pi}{3} - \cot\frac{\pi}{6})$$

$$= -\frac{1}{2}(\frac{\sqrt{3}}{3} - \sqrt{3}) = \frac{\sqrt{3}}{3}.$$

7. $\displaystyle\int \tan \pi x\, dx = \int \tan \pi x\, dx$

$$= \frac{1}{\pi}\ln|\sec\pi x| + C.$$

8. $\displaystyle\int \sec 2z\, dz = \int \frac{1}{\cos 2z}\, dz$

$$= \int \frac{1}{\sin(\frac{\pi}{2} - 2z)}\, dz$$

$$= \frac{1}{2}\int \frac{1}{\sin(\frac{\pi}{4} - z)\cos(\frac{\pi}{4} - z)}\, dz \qquad (u = \frac{\pi}{4} - z)$$

$$= -\frac{1}{2}\int \frac{1}{\tan u \cos^2 u}\, du \qquad (w = \tan u)$$

$$= -\frac{1}{2}\int \frac{1}{w}\, dw$$

$$= -\frac{1}{2}\ln|w| + C$$

$$= -\frac{1}{2}\ln|\tan(\frac{\pi}{4} - z)| + C.$$

9. $\displaystyle\int \sec^2 3x \tan 3x\, dx = \int \frac{1}{\cos^2 3x}\cdot\frac{\sin 3x}{\cos 3x}\, dx \qquad (u = \cos 3x)$

$$= -\frac{1}{3}\int \frac{1}{u^3}\, du$$

$$= \frac{1}{6u^2} + C$$

$$= \frac{1}{6\sin^2 3x} + C$$

10. $\displaystyle\int \cot^3 2x\, dx = \int \frac{\cos^3 2x}{\sin^3 2x}\, dx$

$$= \int \frac{\cos^2 2x \cos 2x}{\sin^3 2x}\, dx$$

$$= \int \frac{(1-\sin^2 2x)\cos 2x}{\sin^3 2x}\, dx \qquad (u = \sin 2x)$$

$$= \frac{1}{2}\int \frac{(1-u^2)}{u^3}\, du$$

$$= \frac{1}{2}\left(-\frac{1}{2u^2} - \ln|u|\right) + C$$

$$= -\frac{1}{4\sin^2 2x} - \frac{\ln|\sin 2x|}{2} + C.$$

11. $$\int_0^{\pi} \tan^3 4x\, dx = \int_0^{\pi} \frac{\sin^3 4x}{\cos^3 4x} dx$$

$$= \int_0^{\pi} \frac{\sin^2 4x \sin 4x}{\cos^3 4x} dx$$

$$= \int_0^{\pi} \frac{(1-\cos^2 4x)\sin 4x}{\cos^3 4x} dx$$

$$= -\frac{1}{4}\left(-\frac{1}{2\cos 4x} - \ln|\cos 4x|\right)\Bigg|_0^{\pi}$$

$$= -\frac{1}{4}\left(-\frac{1}{2} - \ln 1 + \frac{1}{2} + \ln 1\right)\Bigg|_0^{\pi} = 0.$$

12. $$\int \cot 2w \csc^2 2w\, dw = \int \frac{\cos 2w}{\sin 2w}\cdot\frac{1}{\sin^2 2w}\, dw$$

$$= \int \frac{\cos 2w}{\sin^3 2w}\, dw \qquad (u = \sin 2w)$$

$$= \frac{1}{2}\int \frac{1}{u^3}\, du$$

$$= \frac{1}{2}\left(-\frac{1}{2u^2}\right) + C$$

$$= -\frac{1}{4\sin^2 2w} + C.$$

13. $$\int \tan^2 \frac{1}{2}x\, dx = \int\left(\frac{1}{\cos^2 \frac{1}{2}x} - 1\right) dx$$

$$= 2(\tan\frac{1}{2}x - \frac{1}{2}x) + C$$

$$= 2\tan\frac{1}{2}x - x + C$$

14. $\int_{\pi/3}^{\pi/2} \cot^2 2x\, dx = \int_{\pi/3}^{\pi/2} \left(\frac{1}{\sin^2 2x} - 1\right) dx$

$$= (-\frac{\cot 2x}{2} - x)\Big|_{\pi/3}^{\pi/2} = -\infty.$$

15. $\int \frac{\csc^2(\ln x)}{x}\, dx = \int \csc^2 u\, du$ $\quad (u = \ln x)$

$= -\cot u + C$

$= -\cot(\ln x) + C.$

16. $\int x\sec^2(x^2)\, dx = \frac{1}{2}\int \sec^2 u\, du$ $\quad (u = x^2)$

$$= \frac{1}{2}\tan u + C$$

$$= \frac{1}{2}\tan x^2 + C.$$

17. $\int e^x \sec e^x \tan e^x dx = \int \sec u \tan u du$ $\quad (u = e^x)$

$= \sec u + C = \sec e^x + C.$

18. $\int \frac{\tan(\ln x)}{x}\, dx = \int \tan u\, du$ $\quad (u = \ln x)$

$= \ln|\sec u| + C = \ln|\sec(\ln u)| + C.$

19. Let $x = \tan t$, then $dx = \frac{1}{\cos^2 t} dt$.

$$\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{dt}{\cos^2 t(1+\tan^2 t)^{3/2}}$$

$$= \int \frac{dt}{\cos^2 t\left(\frac{1}{\cos^2 t}\right)^{3/2}}$$

$$= \int \cos t dt = \sin t + C = \sin(\tan^{-1} x) + C$$

$$= \frac{x}{\sqrt{1+x^2}} + C.$$

20. Let w = 3sin$t$, then dw = 3cos$t$.

$$\int \frac{\sqrt{9-w^2}}{3\,w}\,dw = \int \frac{\sqrt{9-9\sin^2 t}}{9\sin t}\,3\cos t dt$$

$$= 3\int \frac{\cos^2 t}{\sin t}\,dt = 3\int \frac{1-\sin^2 t}{\sin t}\,dt$$

$$= 3\int (\frac{1}{\sin t} - \sin t)dt$$

$$= \frac{3}{2}\int \frac{1}{\sin\frac{t}{2}\cos\frac{t}{2}}dt - 3\int \sin t dt$$

$$= \frac{3}{2}\int \frac{1}{\tan\frac{t}{2}\cos^2\frac{t}{2}}dt - 3\int \sin t dt$$

$$= 3\ln\left|\tan\frac{t}{2}\right| + 3\cos t + C$$

$$= 3\ln\left|\tan\left(\frac{1}{2}\sin^{-1}\frac{w}{3}\right)\right| - 3\cos\left(\sin^{-1}\frac{w}{3}\right) + C$$

$$= 3\ln\left|\tan\left(\frac{3-\sqrt{9-w^2}}{w}\right)\right| - \sqrt{9-w^2} + C.$$

21. Let $y = \sqrt{2}$ sin$t$, then $dy = \sqrt{2}$ cos$t$.

$$\int \frac{y^2}{\sqrt{2-y^2}}\,dy = \int \frac{2\sin^2 t}{\sqrt{2-2\sin^2 t}}\,\sqrt{2}\cos t dt$$

$$= \int 2\sin^2 t dt = \int 2\sin^2 t dt$$

$$= \int (1-\cos 2t)\,dt$$

$$= t - \frac{1}{2}\sin 2t + C$$

$$= \sin^{-1}\frac{y}{\sqrt{2}} - \frac{1}{2}\sin\left(2\sin^{-1}\frac{y}{\sqrt{2}}\right) + C$$

$$= \sin^{-1}\frac{y}{\sqrt{2}} - \frac{y\sqrt{1-y^2}}{2} + C.$$

22. Let x = 2tan$t$, then $dx = \dfrac{2}{\cos^2 t}dt$.

$$\int \frac{dx}{x^2\sqrt{8+2x^2}} = \int \frac{2dt}{4\tan^2 t\sqrt{8+2\cdot 4\tan^2 t}\,\cos^2 t}$$

$$= \int \frac{dt}{4\sqrt{2}\dfrac{\sin^2 t}{\cos^2 t}\dfrac{1}{\cos t}\cos^2 t}$$

$$= \int \frac{\cos t dt}{4\sqrt{2}\sin^2 t}$$

$$= -\frac{1}{4\sqrt{2}\sin t} + C.$$

$$= -\frac{1}{4\sqrt{2}\sin(\tan^{-1} x/2)} + C$$

$$= -\frac{\sqrt{4+x^2}}{4\sqrt{2}x} + C.$$

23. Let $x = \frac{1}{2}\sin t$, then $dx = \frac{1}{2}\cos t dt$.

$$\int \sqrt{1-4x^2}\,dx = \frac{1}{2}\int \sqrt{1-\sin^2 t}\,\cos t dt$$

$$= \frac{1}{2}\int \cos^2 t dt = \frac{1}{4}\int (1+\cos 2t)dt$$

$$= \frac{1}{4}t + \frac{\sin 2t}{8} + C$$

$$= \frac{1}{4}\sin^{-1} 2x + \frac{\sin(2\sin^{-1} 2x)}{8} + C$$

$$= \frac{1}{4}\sin^{-1} 2x + \frac{x\sqrt{1-4x^2}}{2}.$$

24. Let $x = \frac{5}{2}\tan t$, then $dx = \frac{5}{2\cos^2 t}dt$.

$$\int \frac{dx}{x\sqrt{25+4x^2}} = \int \frac{\dfrac{5}{2}\dfrac{1}{\cos^2 t}dt}{\dfrac{5}{2}\tan t\sqrt{25+25\tan^2 t}}$$

$$= \int \frac{\dfrac{1}{\cos^2 t}dt}{5\dfrac{\sin t}{\cos t}\dfrac{1}{\cos t}}$$

$$= \int \frac{dt}{5\sin t}$$

$$= \frac{1}{5}\int \csc t dt$$

$$= -\frac{1}{5}\ln\left|\csc t + \cot t\right| + C$$

$$= -\frac{1}{5}\ln\left|\csc\left(\tan^{-1}\frac{2}{5}x\right) + \cot\left(\tan^{-1}\frac{2}{5}x\right)\right| + C$$

$$= -\frac{1}{5}\ln\left|\frac{\sqrt{25+4x^2}}{2x} + \frac{5}{2x}\right| + C.$$

25. Let $x = \sin t$, then $dx = \cos t dt$.

$$\int \frac{x\ dx}{\sqrt{1-x^2}} = \int \frac{\sin t \cos t\ dt}{\sqrt{1-\sin^2 t}}$$

$$= \int \frac{\sin t \cos t\ dt}{\cos t}$$

$$= \int \sin t dt = -\cos t + C$$

$$= -\cos(\sin^{-1} x) + C$$

$$= = -\sqrt{1-x^2} + C.$$

26. Let $x = 2\sin t$, then $dx = 2\cos t dt$.

$$\int_0^2 x\sqrt{4-x^2}\ dx = \int_0^{\pi/2} 4\sin t \cos t\sqrt{4-4\sin^2 t}\ dt$$

$$= \int_0^{\pi/2} 8\sin t \cos^2 t\ dt$$

$$= -\frac{8\cos^3 t}{3}\Big|_0^{\pi/2} = -\frac{8}{3}(0-1) = \frac{8}{3}.$$

27. Let $\theta = \sec^{-1}\frac{x}{2}$ then $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta d\theta$, if $x > 2$.

$$\int \frac{\sqrt{x^2-4}}{2x}dx = \int \frac{\sqrt{4\sec^2\theta - 4}}{4\sec\theta}\cdot 2\sec\theta\tan\theta d\theta$$

$$= \int \tan^2\theta d\theta = \int (\sec^2\theta - 1)d\theta$$

$$= \tan\theta - \theta + C =$$

$$= \frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2} + C.$$

28. Let $\theta = \tan^{-1}\dfrac{x}{2\sqrt{3}}$ then $x = 2\sqrt{3}\tan\theta$ and $dx = 2\sqrt{3}\sec^2\theta d\theta$ and secθ is always positive as θ belongs to ($-\pi/2$, $\pi/2$).

$$\int \frac{\sqrt{12+x^2}}{x}dx = \int \frac{\sqrt{12+12\tan^2\theta}}{2\sqrt{3}\tan\theta}\cdot 2\sqrt{3}\sec^2\theta d\theta$$

$$= \int \frac{2\sqrt{3}\sec^3\theta}{\tan\theta}d\theta = 2\sqrt{3}\int \frac{1}{\sin\theta\cos^2\theta}d\theta$$

$$= 2\sqrt{3}\int\left(\frac{\sin\theta}{\cos^2\theta}+\frac{1}{\sin\theta}\right)d\theta$$

$$= 2\sqrt{3}\left(\sec\theta - \ln(\csc\theta+\cot\theta)\right)+C$$

$$= 2\sqrt{3}\left(\frac{\sqrt{12+x^2}}{2\sqrt{3}} - \ln(\frac{\sqrt{12+x^2}}{x}+\frac{2\sqrt{3}}{x})\right)+C$$

$$= \sqrt{12+x^2} - 2\sqrt{3}\ln\left(\frac{\sqrt{12+x^2}}{x}+\frac{2\sqrt{3}}{x}\right)+C$$

29. Let $\theta = \tan^{-1}\dfrac{x}{4}$ then x = 4tanθ and $dx = 4\sec^2\theta d\theta$ and secθ is always positive as θ belongs to (0, $\tan^{-1}\dfrac{3}{4}$).

$$\int_0^3 \frac{dx}{\sqrt{16+x^2}} = \int_0^{\tan^{-1}\frac{3}{4}} \frac{4\sec^2\theta d\theta}{\sqrt{16+16\tan^2\theta}}$$

$$= \int_0^{\tan^{-1}\frac{3}{4}} \frac{\sec^2\theta d\theta}{\sec\theta} = \int_0^{\tan^{-1}\frac{3}{4}} \sec\theta d\theta$$

$$= \ln(\sec\theta+\tan\theta)\Big|_0^{\tan^{-1}\frac{3}{4}}$$

$$= \ln(\frac{\sqrt{9+16}}{16}+\frac{3}{4}) - \ln(1+0)$$

= ln2.

30. Let $\theta = \tan^{-1}\dfrac{3x}{4}$ then $x = \dfrac{4}{3}$ tanθ and $dx = \dfrac{4}{3}\sec^2\theta d\theta$ and secθ is always positive as θ belongs to ($-\pi/2$, $\pi/2$).

$$\int x^2\sqrt{16+9x^2}dx = \int \frac{16}{9}\tan^2\theta\sqrt{16+16\tan^2\theta}\cdot\frac{4}{3}\sec^2\theta d\theta = \frac{256}{27}\int \tan^2\theta\sec^3\theta d\theta$$

$$= \frac{256}{27}\int (\sec^5\theta - \sec^3\theta)d\theta \quad =$$

$$= \frac{256}{27}\int\left(\frac{1}{\sin^5(\frac{\pi}{2}-\theta)} - \frac{1}{\sin^3(\frac{\pi}{2}-\theta)}\right)d\theta$$

$$= \frac{256}{27}\int\left(\frac{1}{32\sin^5(\frac{\pi}{4}-\frac{\theta}{2})\cos^5(\frac{\pi}{4}-\frac{\theta}{2})} - \frac{1}{8\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta$$

$$= \frac{256}{27}\int\left(\frac{\left(\sin^2(\frac{\pi}{4}-\frac{\theta}{2})+\cos^2(\frac{\pi}{4}-\frac{\theta}{2})\right)^3}{32\sin^5(\frac{\pi}{4}-\frac{\theta}{2})\cos^5(\frac{\pi}{4}-\frac{\theta}{2})} - \frac{\sin^2(\frac{\pi}{4}-\frac{\theta}{2})+\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}{4\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta$$

$$=$$

$$\frac{256}{27}\int\left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{32\cos^5(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{3}{32\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{3}{32\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{32\sin^5(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta$$

$$-\frac{256}{27}\int\left(\frac{1}{4\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{1}{4\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta \; =$$

$$\frac{256}{27}\int\left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{32\cos^5(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{32\sin^5(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{3}{32}\left(\frac{2}{\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{\sin^3(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}\right)\right)d\theta$$

$$-\frac{256}{27}\int\left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{4\cos^3(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{2}{4\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{4\sin^3(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta \; =$$

$$\frac{256}{27}\left(\frac{2}{32(-4)\cos^4(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{-2}{32(-4)\sin^4(\frac{\pi}{4}-\frac{\theta}{2})}\right) +$$

$$\frac{256}{27}\left(\frac{3\cdot 4}{32}\ln\left|\sin(\frac{\pi}{2}-\theta)\right| + \frac{3}{32}\cdot\frac{(-2)}{(-2)\sin^2(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{3}{32}\cdot\frac{2}{(-2)\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}\right) -$$

$$\frac{-2}{4}\cdot\frac{256}{27}\left(\frac{-1}{(-2)\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{1}{(-2)\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}\right)-\ln\left|\sin(\frac{\pi}{2}-\theta)\right|+C =$$

$$\frac{2}{27}\left(\frac{\cos^2(\frac{\pi}{4}-\frac{\theta}{2})-\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}{\cos^4(\frac{\pi}{4}-\frac{\theta}{2})\sin^4(\frac{\pi}{4}-\frac{\theta}{2})}\right)+\left(\frac{32}{9}\ln\left|\sin(\frac{\pi}{2}-\theta)\right|+\frac{8}{9}\frac{1}{\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}-\frac{8}{9}\frac{1}{\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}\right)-$$

$$\frac{256}{27}\left(\frac{-1}{4\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{1}{4\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}-\ln|\cos\theta|\right)+C$$

$$=\frac{2}{27}\left(\frac{64\cos(\frac{\pi}{2}-\theta)}{\sin^4(\frac{\pi}{2}-\theta)}\right)+\left(\frac{32}{9}\ln|\cos\theta|+\frac{8}{9}\cdot\frac{4\cos(\frac{\pi}{2}-\theta)}{\sin^2(\frac{\pi}{2}-\theta)}\right)-\frac{256}{27}\left(\frac{\sin\theta}{\cos^2\theta}-\ln|\cos\theta|\right)+C$$

$$=\frac{2}{27}\left(\frac{64\sin\theta}{\cos^4\theta}\right)+\left(\frac{32}{9}\ln|\cos\theta|+\frac{8}{9}\cdot\frac{4\sin\theta}{\cos^2\theta}\right)-\frac{256}{27}\left(\frac{\sin\theta}{\cos^2\theta}-\ln|\cos\theta|\right)+C$$

$$=\frac{2}{27}\left(64\frac{\sqrt{x^2-2}}{x}\frac{x^4}{2}\right)+\left(\frac{32}{9}\ln\left|\cos\frac{\sqrt{2}}{x}\right|+\frac{8}{9}\cdot\frac{4\sqrt{x^2-2}}{x}\frac{x^2}{2}\right)-\frac{256}{27}\left(\frac{\sqrt{x^2-2}}{x}\frac{x^2}{2}-\ln\left|\cos\frac{\sqrt{2}}{x}\right|\right)+C$$

$$=\frac{64x^3\sqrt{x^2-2}}{27}+\frac{32}{9}\ln\left|\cos\frac{\sqrt{2}}{x}\right|+\frac{16}{9}x\sqrt{x^2-2}-\frac{128}{27}x\sqrt{x^2-2}-\frac{256}{27}\ln\left|\cos\frac{\sqrt{2}}{x}\right|+C$$

$$=\frac{64x^3\sqrt{x^2-2}}{27}+\frac{32}{9}\ln\left|\cos\frac{\sqrt{2}}{x}\right|-\frac{80}{27}x\sqrt{x^2-2}-\frac{256}{27}\ln\left|\cos\frac{\sqrt{2}}{x}\right|+C.$$

31. Let $\theta = \tan^{-1}\frac{3x}{2}$ then $x = \frac{2}{3}\tan\theta$ and $dx = \frac{2}{3}\sec^2\theta d\theta$ and $\sec\theta$ is always positive as θ belongs to $(-\pi/2, \pi/2)$.

$$\int\frac{dx}{x^2\sqrt{4+9x^2}} = \int\frac{9\cdot 2\sec^2\theta d\theta}{4\cdot 3\tan^2\theta\sqrt{4+4\tan^2\theta}}$$

$$= \int\frac{3\sec\theta d\theta}{4\tan^2\theta} = \int\frac{3\cos\theta d\theta}{4\sin^2\theta}$$

$$= -\frac{3}{4\sin\theta}+C = -\frac{3}{4}\csc\theta+C$$

$$= -\frac{3}{4}\frac{\sqrt{4+9x^2}}{3x}+C = -\frac{\sqrt{4+9x^2}}{4}+C.$$

32. Let $\theta = \sec^{-1}\left|\frac{11y}{2}\right|$ then $|y| = \frac{2}{11}\sec\theta$ and $dy = \frac{2}{11}\sec\theta\tan\theta d\theta$ if $y \ge \frac{2}{11}$ and $dy =$ $-\frac{2}{11}\sec\theta\tan\theta d\theta$ if $y \le -\frac{2}{11}$.

$$\int \frac{dy}{y\sqrt{121y^2-4}} = \begin{cases} \displaystyle\int \frac{2\cdot 11\sec\theta\tan\theta d\theta}{11\cdot 2\sec\theta\sqrt{4\sec^2\theta-4}}, y \ge \frac{2}{11} \\ \displaystyle\int \frac{-2\cdot 11\sec\theta\tan\theta d\theta}{-11\cdot 2\sec\theta\sqrt{4\sec^2\theta-4}}, y \le -\frac{2}{11} \end{cases}$$

$$= \int \frac{\tan\theta d\theta}{2\tan\theta} = \frac{1}{2}\theta + C = \frac{1}{2}\sec^{-1}\left|\frac{11y}{2}\right| + C$$

33. Let $\theta = \sec^{-1}\left|\frac{2x}{3}\right|$ then $|x| = \frac{3}{2}\sec\theta$ and $dx = \frac{3}{2}\sec\theta\tan\theta d\theta$ if $x > 3$.

$$\int \frac{dx}{x^2\sqrt{4x^2-9}} = \int \frac{3/2\cdot\sec\theta\tan\theta d\theta}{3/2\cdot\sec^2\theta\sqrt{9\sec^2\theta-9}}$$

$$= \int \frac{\tan\theta d\theta}{\sec\theta\tan\theta} = \int \cos\theta d\theta = \sin\theta + C = \frac{\sqrt{4x^2-9}}{2x} + C.$$

34. Let $\theta = \tan^{-1}\frac{x}{3}$ then $x = 3\tan\theta$ and $dx = 3\sec^2\theta d\theta$ and $\sec\theta$ is always positive as θ belongs to $(-\pi/2, \pi/2)$.

$$\int_0^4 \frac{x^2}{\sqrt{9+x^2}}dx = \int_0^{\tan^{-1}\frac{3}{4}} \frac{9\tan^2\theta}{\sqrt{9+9\tan^2\theta}}3\sec^2\theta d\theta$$

$$= \int_0^{\tan^{-1}\frac{3}{4}} 9\tan^2\theta\sec\theta d\theta$$

$$= 9\int_0^{\tan^{-1}\frac{3}{4}} (\sec^3\theta - \sec\theta)d\theta$$

$$= 9\int_0^{\tan^{-1}\frac{3}{4}} (\sec^3\theta - \sec\theta)d\theta$$

$$= 9\int_0^{\tan^{-1}\frac{3}{4}} \frac{1}{\sin^3(\frac{\pi}{2}-\theta)}d\theta - 9\int_0^{\tan^{-1}\frac{3}{4}} \sec\theta d\theta$$

$$= 9\int_0^{\tan^{-1}\frac{3}{4}} \frac{1}{8\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}d\theta - 9\int_0^{\tan^{-1}\frac{3}{4}} \sec\theta d\theta$$

$$= 9\int_0^{\tan^{-1}\frac{3}{4}} \frac{\sin^2(\frac{\pi}{4}-\frac{\theta}{2})+\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}{8\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}d\theta - 9\int_0^{\tan^{-1}\frac{3}{4}} \sec\theta d\theta$$

$$= 9\int_0^{\tan^{-1}\frac{3}{4}} \left(\frac{1}{8\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{1}{8\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta - 9\int_0^{\tan^{-1}\frac{3}{4}} \sec\theta d\theta$$

$$= \frac{9}{8}\int_0^{\tan^{-1}\frac{3}{4}} \left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{2}{\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{\sin^3(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta - 9\int_0^{\tan^{-1}\frac{3}{4}} \sec\theta d\theta$$

$$=$$

$$9\left(\frac{-1\cdot(-2)}{(-2)\cdot 8\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{1}{2}\ln\left|\sin(\frac{\pi}{2}-\theta)\right|+\frac{(-2)}{(-2)\cdot 8\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}-\ln|\sec\theta+\tan\theta|\right)\Bigg|_0^{\tan^{-1}\frac{3}{4}}$$

$$= \left(\frac{9}{8}\frac{4\sin\theta}{\cos^2\theta}+\frac{1}{2}\ln\left|\cos\theta\right|-\ln|\sec\theta+\tan\theta|\right)\Bigg|_0^{\tan^{-1}\frac{3}{4}}$$

$$= \frac{9}{8}\cdot 4\cdot\frac{3}{5}\cdot\frac{25}{16}+\frac{1}{2}\ln\frac{4}{5}-\ln\frac{3}{8}$$

$$= \frac{135}{32}+\frac{1}{2}\ln\frac{4}{5}-\ln\frac{3}{8}.$$

35. Let $\theta = \sin^{-1}\frac{x}{2}$ then $x = 2\sin\theta$ and $dx = 2\cos\theta d\theta$ if $0 < x < 2$.

$$\int_0^2 x^2\sqrt{4-x^2}\ dx = \int_0^{\pi/2} 4\sin^2\theta\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta \mathrm{d}\theta$$

$$= \int_0^{\pi/2} 16\sin^2\theta\cos^2\theta \mathrm{d}\theta$$

$$= 4\int_0^{\pi/2} (2\sin\theta\cos\theta)^2 \mathrm{d}\theta$$

$$= 4\int_0^{\pi/2} \sin^2 2\theta \mathrm{d}\theta$$

$$= 2\int_0^{\pi/2} (1-\cos 4\theta)\mathrm{d}\theta$$

$$= \left(2\theta - \frac{\sin 4\theta}{2}\right)\Bigg|_0^{\pi/2} = \pi.$$

36. Let $\theta = \sec^{-1}\dfrac{x}{2\sqrt{2}}$ then $x = 2\sqrt{2}\sec\theta$ and $dx = 2\sqrt{2}\sec\theta\tan\theta d\theta$, if $x > 2\sqrt{2}$.

$$\int \frac{\sqrt{x^2-8}\ dx}{x^2} = \int \frac{\sqrt{8\sec^2\theta - 8}}{8\sec^2\theta}\cdot 2\sqrt{2}\sec\theta\tan\theta d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta} d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta} d\theta = \int\left(\frac{1}{\cos\theta} - \cos\theta\right) d\theta$$

$= \ln|\sec\theta + \tan\theta| - \sin\theta + C =$

$$= \ln\left|\frac{x}{2\sqrt{2}} + \frac{\sqrt{x^2-8}}{8}\right| - \frac{\sqrt{x^2-8}}{x} + C.$$

37. Let $\theta = \sin^{-1}\dfrac{s}{3}$ then $s = 3\sin\theta$ and $ds = 3\cos\theta d\theta$ and $\cos\theta$ is always positive as θ belongs to $[-\pi/2, \pi/2]$.

$$\int \frac{\sqrt{9-s^2}}{s^2}\, ds = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} 3\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

$$= \int\left(\csc^2\theta - 1\right) d\theta$$

$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-s^2}}{s} - \sin^{-1}\frac{s}{3} + C$$

38. Let $\theta = \sin^{-1}\dfrac{x}{3}$ then $x = 3\sin\theta$ and $dx = 3\cos\theta d\theta$ and $\cos\theta$ is always positive as θ belongs to $[-\pi/2, \pi/2]$.

$$\int_0^3 x^3\sqrt{9-x^2}\ dx = \int_0^{\pi/2} 27\sin^3\theta\sqrt{9-9\sin^2\theta}\ 3\cos\theta d\theta$$

$$= \int_0^{\pi/2} 81\sin^3\theta\ \cos^2\theta d\theta$$

$$= 81\int_0^{\pi/2} (1-\cos^2\theta)\cos^2\theta\sin\theta d\theta$$

$$= 81\int_0^{\pi/2} (\cos^2\theta-\cos^4\theta)\sin\theta d\theta$$

$$= -81\left(\frac{\cos^3\theta}{3}-\frac{\cos^5\theta}{5}\right)\Bigg|_0^{\pi/2}$$

$$= 0+81\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{54}{5}.$$

39. Let $\theta = \sec^{-1}\dfrac{x}{\sqrt{2}}$ then $x=\sqrt{2}\sec\theta$ and $dx = \sqrt{2}\sec\theta\tan\theta d\theta$, if $x > \sqrt{2}$.

$$\int\frac{x^2}{\sqrt{x^2-2}}dx = \int\frac{2\sec^2\theta}{\sqrt{2\sec^2\theta-2}}\cdot\sqrt{2}\sec\theta\tan\theta d\theta$$

$$= \int\frac{2\sqrt{2}\sec^3\theta}{\sqrt{2}\tan\theta}\tan\theta d\theta$$

$$= 2\int\sec^3\theta\, d\theta = 2\int\frac{1}{\cos^3\theta}d\theta$$

$$= 2\int\frac{1}{\sin^3(\frac{\pi}{2}-\theta)}d\theta = 2\int\frac{1}{8\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}d\theta$$

$$= \int\frac{\sin^2(\frac{\pi}{4}-\frac{\theta}{2})+\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}{4\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}d\theta$$

$$= \int\left(\frac{1}{4\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{1}{4\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta$$

$$= \int\left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{4\cos^3(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{2}{4\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{4\sin^3(\frac{\pi}{4}-\frac{\theta}{2})}\right)d\theta$$

$$= \frac{-2}{4}\left(\frac{-1}{(-2)\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{1}{(-2)\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}\right)-\ln\left|\sin(\frac{\pi}{2}-\theta)\right|+C$$

$$= \frac{-1}{4\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}+\frac{1}{4\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}-\ln\left|\cos\theta\right|+C$$

$$= \frac{\sin\theta}{\cos^2\theta}-\ln\left|\cos\theta\right|+C$$

$$= \frac{\sqrt{x^2-2}}{x}\frac{x^2}{2}-\ln\left|\cos\frac{\sqrt{2}}{x}\right|+C$$

$$= \frac{x\sqrt{x^2-2}}{2}-\ln\left|\cos\frac{\sqrt{2}}{x}\right|+C\,.$$

40. Let $\theta = \sin^{-1} x$ then $x = \sin\theta$ and $dx = \cos\theta d\theta$ and $\cos\theta$ is always positive as θ belongs to $[0, \pi/2]$.

$$\int_0^1 x^2\sqrt{1-x^2}\ dx = \int_0^{\pi/2} \sin^2\theta\sqrt{1-\sin^2\theta}\ \cos\theta d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\ \cos^2\theta d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^2 2\theta}{4}\ d\theta$$

$$= \int_0^{\pi/2} \frac{1-\cos 4\theta}{4}\ d\theta$$

$$= \left(\frac{\theta}{4}-\frac{\cos 4\theta}{16}\right)\Bigg|_0^{\pi/2}$$

$$= \frac{\pi}{8}-\frac{1}{16}-0+\frac{1}{16} = \frac{\pi}{8}.$$

41. Let $\theta = \sin^{-1}\frac{s}{3\sqrt{2}}$ then $s = 3\sqrt{2}\sin\theta$ and $ds = 3\sqrt{2}\cos\theta d\theta$ and $\cos\theta$ is always positive as θ belongs to $[-\pi/2, \pi/2]$.

$$\int \frac{s^2}{\sqrt{18-s^2}}\ ds = \int \frac{18\sin^2\theta}{\sqrt{18-18\sin^2\theta}}\cdot 3\sqrt{2}\cos\theta d\theta$$

$$= \int 18\sin^2\theta d\theta = 9\int(1-\cos 2\theta)d\theta$$

$$= 9(\theta - \frac{\sin 2\theta}{2}) + C$$

$$= 9\theta - \frac{9}{2}\sin\theta\cos\theta + C$$

$$= 9\sin^{-1}\frac{s}{3\sqrt{2}} - \frac{9}{2}\frac{s}{3\sqrt{2}}\frac{\sqrt{18-s^2}}{3\sqrt{2}} + C$$

$$= 9\sin^{-1}\frac{s}{3\sqrt{2}} - \frac{s\sqrt{18-s^2}}{4} + C\,.$$

42. Let $\theta = \sin^{-1}\frac{r}{\sqrt{2}}$ then $r = \sqrt{2}\sin\theta$ and $dr = \sqrt{2}\cos\theta d\theta$ and $\cos\theta$ is always positive as θ belongs to $[-\pi/2, \pi/2]$.

$$\int \frac{1}{r\sqrt{2-r^2}}\,dr = \int \frac{\sqrt{2}\cos\theta}{\sqrt{2}\sin\theta\sqrt{2-2\sin^2\theta}}\,d\theta$$

$$= \int \frac{d\theta}{\sin\theta}$$

$$= -\ln|\csc\theta + \cot\theta| + C$$

$$= -\ln\left|\frac{\sqrt{2}}{r} + \frac{\sqrt{2-r^2}}{r}\right| + C.$$

43. Let $\theta = \sec^{-1}x$ then $x = \sec\theta$ and $dx = \sec\theta\tan\theta d\theta$, if $\sqrt{2} < x < \sqrt{3}$.

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{x^2}{\left(x^2-1\right)^{5/2}}\,dx = \int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \frac{\sec^2\theta}{\left(\sec^2\theta-1\right)^{5/2}}\sec\theta\tan\theta \mathrm{d}\theta$$

$$= \int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \frac{\sec^3\theta}{\tan^4\theta}\,\mathrm{d}\theta$$

$$= \int_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}} \frac{\cos\theta}{\sin^4\theta}\,\mathrm{d}\theta$$

$$= -\frac{1}{3\sin^3\theta}\Bigg|_{\pi/4}^{\cos^{-1}\frac{1}{\sqrt{3}}}$$

$$= -\frac{1}{3}(\frac{3\sqrt{3}}{2\sqrt{2}} - \frac{8}{2\sqrt{2}})$$

$$= \frac{8-3\sqrt{3}}{6\sqrt{2}}.$$

44. Let $\theta = \sec^{-1}\frac{x}{2}$ then $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta d\theta$, if $\sqrt{5} < x < \sqrt{6}$.

$$\int_{\sqrt{5}}^{\sqrt{6}} \frac{1}{\left(x^2-4\right)^{3/2}}\, dx = \int_{\sec^{-1}\frac{\sqrt{5}}{2}}^{\sec^{-1}\frac{\sqrt{6}}{2}} \frac{2\sec\theta\tan\theta d\theta}{\left(4\sec^2\theta-4\right)^{3/2}}$$

$$= \int_{\sec^{-1}\frac{\sqrt{5}}{2}}^{\sec^{-1}\frac{\sqrt{6}}{2}} \frac{2\sec\theta\tan\theta d\theta}{8\tan^3\theta}$$

$$= \int_{\sec^{-1}\frac{\sqrt{5}}{2}}^{\sec^{-1}\frac{\sqrt{6}}{2}} \frac{\sec\theta\, d\theta}{4\tan^2\theta} = \int_{\sec^{-1}\frac{\sqrt{5}}{2}}^{\sec^{-1}\frac{\sqrt{6}}{2}} \frac{\cos\theta\, d\theta}{4\sin^2\theta}$$

$$= -\frac{1}{4\sin\theta}\Bigg|_{\sec^{-1}\frac{\sqrt{5}}{2}}^{\sec^{-1}\frac{\sqrt{6}}{2}} = -\frac{1}{4}\csc\theta\Bigg|_{\sec^{-1}\frac{\sqrt{5}}{2}}^{\sec^{-1}\frac{\sqrt{6}}{2}}$$

$$= \frac{\sqrt{5}-\sqrt{3}}{4}.$$

45. Let $\theta = \sin^{-1}\frac{r}{2}$ then $r = 2\sin\theta$ and $dr = 2\cos\theta d\theta$ and $\cos\theta$ is always positive as θ belongs to $[0, \pi/2]$.

$$\int_0^1 \frac{1}{\left(4-r^2\right)^{3/2}}\, dr = \int_0^{\pi/6} \frac{1}{\left(4-4\sin^2\theta\right)^{3/2}}\, 2\cos\theta d\theta$$

$$= \int_0^{\pi/6} \frac{2\cos\theta}{8\cos^3\theta}\, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{4}\sec^2\theta d\theta$$

$$= \frac{1}{4}\tan\theta\Bigg|_0^{\pi/6}$$

$$= \frac{1}{4\sqrt{3}}.$$

46. Let $\theta = \sec^{-1} e^x$ then $x = \ln(\sec\theta)$ and $dx = \dfrac{\sec\theta\,\tan\theta d\theta}{\sec\theta}$ = $\tan\theta d\theta$ as $x > 0$.

$$\int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx = \int \frac{\sec\theta}{\sqrt{\sec^2\theta-1}}\tan\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|e^x + \sqrt{e^{2x}-1}| + C.$$

47. Let $\theta = \sin^{-1}(\ln x)$ then $\ln x = \sin\theta$, $x = e^{\sin\theta}$ and $dx = e^{\sin\theta}\cos\theta d\theta$ as $x > 0$ and $x \neq 1$.

$$\int \frac{\sqrt{1-(\ln x)^2}}{x(\ln x)^2}\,dx = \int \frac{\sqrt{1-\sin^2\theta}}{e^{\sin\theta}\sin^2\theta}\,e^{\sin\theta}\cos\theta d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta$$

$$= -\cot\theta - \theta + C = -\frac{\sqrt{1-\ln^2 x}}{\ln x} - \sin^{-1}(\ln x) + C.$$

48. Let $\theta = \tan^{-1}\dfrac{2x}{7}$ then $x = \dfrac{7}{2}\tan\theta$ and $dx = \dfrac{7}{2}\sec^2\theta d\theta$ and $\sec\theta$ is always positive as θ belongs to $(-\pi/2, \pi/2)$.

$$\int \frac{\sqrt{49+4x^2}}{x}dx = \int \frac{\sqrt{49+49\tan^2\theta}}{7/2\tan\theta}\cdot\frac{7}{2}\sec^2\theta d\theta$$

$$= \int \frac{7\sec^3\theta}{\tan\theta}d\theta$$

$$= 7\int \frac{1}{\cos^2\theta\sin\theta}d\theta$$

$$= 7\int\left(\frac{1}{\sin\theta} + \frac{\sin\theta}{\cos^2\theta}\right)d\theta$$

$$= -7\ln|\csc\theta + \cot\theta| + \frac{7}{\cos\theta} + C$$

$$= -7\ln\left|\frac{\sqrt{49+4x^2}}{2x} + \frac{7}{2x}\right| + \sqrt{49+4x^2} + C$$

49. Let $\theta = \sin^{-1} x^2$ then $x = \sqrt{\sin\theta}$ and $dx = \dfrac{\cos\theta d\theta}{2\sqrt{\sin\theta}}$ and $\cos\theta$ is always positive as θ belongs to $[-\pi/2, \pi/2]$.

$$\int \frac{\sqrt{1-x^4}}{x^3}\,dx = \int \frac{\sqrt{1-\sin^2\theta}}{(\sqrt{\sin\theta})^3}\,\frac{\cos\theta d\theta}{2\sqrt{\sin\theta}}$$

$$= \int \frac{\cos^2\theta d\theta}{2\sin^2\theta}$$

$$= \frac{1}{2}\int \frac{(1-\sin^2\theta)d\theta}{\sin^2\theta}$$

$$= \frac{1}{2}(-\cot\theta-\theta)+C$$

$$= -\frac{\sqrt{1-x^4}}{2x^2}-\frac{1}{2}\sin^{-1}x^2+C\,.$$

50. Let $\theta = \sin^{-1}(\ln x)$ then ln$x$ = sinθ, x = $e^{\sin\theta}$ and dx = $e^{\sin\theta}\cos\theta d\theta$ as x >0 and $x \neq 1$.

$$\int \frac{\sqrt{1-(\ln x)^2}}{x(\ln x)^2}\,dx = \int \frac{\sqrt{1-\sin^2\theta}}{e^{\sin\theta}\sin^2\theta}\,e^{\sin\theta}\cos\theta d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta$$

$$= -\cot\theta-\theta+C = -\frac{\sqrt{1-\ln^2 x}}{\ln x}-\sin^{-1}(\ln x)+C\,.$$

51. Let $\theta = \sin^{-1}x^2$ then x = $\sqrt{\sin\theta}$ and dx = $\dfrac{\cos\theta\mathrm{d}\theta}{2\sqrt{\sin\theta}}$ and cosθ is always positive as θ belongs to [$-\pi$/2, π/2].

$$\int \frac{\sqrt{1-x^4}}{x^3}\,dx = \int \frac{\sqrt{1-\sin^2\theta}}{(\sqrt{\sin\theta})^3}\,\frac{\cos\theta\mathrm{d}\theta}{2\sqrt{\sin\theta}}$$

$$= \int \frac{\cos^2\theta d\theta}{2\sin^2\theta}$$

$$= \frac{1}{2}\int \frac{(1-\sin^2\theta)d\theta}{\sin^2\theta}$$

$$= \frac{1}{2}(-\cot\theta-\theta)+C$$

$$= -\frac{\sqrt{1-x^4}}{2x^2}-\frac{1}{2}\sin^{-1}x^2+C\,.$$

52. Let $\theta = \sec^{-1}x$ then $x=\sec\theta$ and dx = secθ tan$\theta d\theta$, if 1 < x < 2.

$$\int_1^2 x^2\sqrt{x^2-1}\,dx = \int_0^{\pi/3} \sec^2\theta\sqrt{\sec^2\theta-1}\,\sec\theta\,\tan\theta d\theta$$

$$= \int_0^{\pi/3} \sec^3\theta\tan^2\theta\,d\theta$$

$$= \int_0^{\pi/3} (\sec^5\theta-\sec^3\theta)\,d\theta$$

$$= \int_0^{\pi/3} (\sec^5\theta - \sec^3\theta)\, d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{\sin^5(\pi/2-\theta)} - \frac{1}{\sin^3(\pi/2-\theta)} \right) d\theta$$

$$= \int_0^{\pi/3} \left(\frac{1}{32\sin^5(\pi/4-\theta/2)\cos^5(\pi/4-\theta/2)} - \frac{1}{8\sin^3(\pi/4-\theta/2)\cos^3(\pi/4-\theta/2)} \right) d\theta$$

$$= \int_0^{\pi/3} \left(\frac{\left(\sin^2(\pi/4-\theta/2)+\cos^2(\pi/4-\theta/2)\right)^3}{32\sin^5(\pi/4-\theta/2)\cos^5(\pi/4-\theta/2)} - \frac{\sin^2(\pi/4-\theta/2)+\cos^2(\pi/4-\theta/2)}{4\sin^3(\pi/4-\theta/2)\cos^3(\pi/4-\theta/2)} \right) d\theta$$

$$= \int_0^{\pi/3} \left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{32\cos^5(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{3}{32\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{3}{32\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{32\sin^5(\frac{\pi}{4}-\frac{\theta}{2})} \right) d\theta$$

$$- \int_0^{\pi/3} \left(\frac{1}{4\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos^3(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{1}{4\sin^3(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})} \right) d\theta =$$

$$\int_0^{\pi/3} \left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{32\cos^5(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{32\sin^5(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{3}{32}\left(\frac{2}{\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{\sin^3(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{\cos^3(\frac{\pi}{4}-\frac{\theta}{2})} \right) \right) d\theta$$

$$- \int_0^{\pi/3} \left(\frac{\sin(\frac{\pi}{4}-\frac{\theta}{2})}{4\cos^3(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{2}{4\sin(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{\cos(\frac{\pi}{4}-\frac{\theta}{2})}{4\sin^3(\frac{\pi}{4}-\frac{\theta}{2})} \right) d\theta =$$

$$\left(\frac{2}{32(-4)\cos^4(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{-2}{32(-4)\sin^4(\frac{\pi}{4}-\frac{\theta}{2})} \right)\Bigg|_0^{\pi/3} +$$

$$\left(\frac{3\cdot 4}{32}\ln\left|\sin(\frac{\pi}{2}-\theta)\right| + \frac{3}{32}\cdot\frac{(-2)}{(-2)\sin^2(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{3}{32}\cdot\frac{2}{(-2)\cos^2(\frac{\pi}{4}-\frac{\theta}{2})} \right)\Bigg|_0^{\pi/3} -$$

$$\frac{-2}{4}\left(\frac{-1}{(-2)\cos^2(\frac{\pi}{4}-\frac{\theta}{2})} + \frac{1}{(-2)\sin^2(\frac{\pi}{4}-\frac{\theta}{2})} \right)\Bigg|_0^{\pi/3} - \ln\left|\sin(\frac{\pi}{2}-\theta)\right|\Bigg|_0^{\pi/3} =$$

$$\left(\frac{\cos^2(\frac{\pi}{4}-\frac{\theta}{2})-\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}{64\cos^4(\frac{\pi}{4}-\frac{\theta}{2})\sin^4(\frac{\pi}{4}-\frac{\theta}{2})}\right)\Bigg|_0^{\pi/3} + \left(\frac{3\cdot 4}{32}\ln|\cos\theta| + \frac{3}{32}\cdot\frac{\cos^2(\frac{\pi}{4}-\frac{\theta}{2})-\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}{\cos^2(\frac{\pi}{4}-\frac{\theta}{2})\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}\right)\Bigg|_0^{\pi/3} -$$

$$\frac{-1}{4}\left(\frac{\sin^2(\frac{\pi}{4}-\frac{\theta}{2})-\cos^2(\frac{\pi}{4}-\frac{\theta}{2})}{\cos^2(\frac{\pi}{4}-\frac{\theta}{2})\sin^2(\frac{\pi}{4}-\frac{\theta}{2})}\right)\Bigg|_0^{\pi/3} - \ln|\cos\theta|\Big|_0^{\pi/3} = \left(\frac{\sin\theta}{4\cos^4\theta}\right)\Bigg|_0^{\pi/3} + \frac{3}{8}\left(\ln|\cos\theta| + \frac{\sin\theta}{\cos^2\theta}\right)\Bigg|_0^{\pi/3} -$$

$$\left(\frac{\sin\theta}{\cos^2\theta}\right)\Bigg|_0^{\pi/3} - \ln|\cos\theta|\Big|_0^{\pi/3} = \frac{3}{4}\sqrt{3} - \frac{5}{8}\ln 2 .$$

53. i- The are of the sector of the circle is $S_1 = \frac{a^2\theta}{2}$.

The area of the triangle is $S_2 = \frac{ab\sin\theta}{2}$.

Then the are of the shaded region is $S = S_1 - S_2 = \frac{a^2\theta - ab\sin\theta}{2}$.

ii- $S = \int_b^a \sqrt{a^2 - x^2}\,dx$.

Using a trigonometric substitution $\theta = \sin^{-1} x$
we have

$$\int_b^a \sqrt{a^2-x^2}\,dx = \int_{\sin^{-1}b}^{\sin^{-1}a} \sqrt{a^2 - a^2\sin^2\theta}\cdot a\cos\theta d\theta = \int_{\sin^{-1}b}^{\sin^{-1}a} a^2\cos^2\theta d\theta = \frac{a^2}{2}\int_{\sin^{-1}b}^{\sin^{-1}a}(1-\cos 2\theta)d\theta =$$

$$\frac{a^2}{2}(\theta - \frac{\sin 2\theta}{2})\Bigg|_{\sin^{-1}b}^{\sin^{-1}a} = \frac{a^2}{2}(\theta - \sin\theta\cos\theta)\Big|_{\sin^{-1}b}^{\sin^{-1}a} = \frac{a^2}{2}(\sin^{-1}a - \sin^{-1}b - a\sqrt{1-a^2} + b\sqrt{1-b^2}) .$$

2. Partial Fractions

1. $\dfrac{-1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1} = \dfrac{Ax-A+Bx}{x(x-1)} = \dfrac{(A+B)x-A}{x(x-1)}$.

This implies that $-1 = (A + B)x - A$.

$$\begin{cases} A+B=0 \\ -A=-1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \end{cases}.$$

Therefore, $\dfrac{-1}{x(x-1)} = \dfrac{1}{x} - \dfrac{1}{x-1}$

2. $\dfrac{4x}{x^2-1} = \dfrac{4x}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} = \dfrac{A(x+1)+B(x-1)}{(x-1)(x+1)} = \dfrac{(A+B)x+A-B}{(x-1)(x+1)}$.

This implies that $4x = (A + B)x + A - B$.

$$\begin{cases} A+B=4 \\ A-B=0 \end{cases} \Rightarrow \begin{cases} B=2 \\ A=2 \end{cases}.$$

Therefore, $\dfrac{4x}{x^2-1} = \dfrac{2}{x-1} - \dfrac{2}{x+1}$.

3. $\dfrac{8x-8}{x^2+6x-55} = \dfrac{8x-8}{(x+11)(x-5)} = \dfrac{A}{x+11} + \dfrac{B}{x-5} = \dfrac{A(x-5)+B(x+11)}{(x+11)(x-5)} =$

$\dfrac{(A+B)x-5A+11B}{(x+11)(x-5)}$.

This implies that $8x - 8 = (A + B)x - 5A + 11B$.

$$\begin{cases} A+B=8 \\ -5A+11B=-8 \end{cases} \Rightarrow \begin{cases} B=2 \\ A=6 \end{cases}.$$

Therefore, $\dfrac{8x-8}{x^2+6x-55} = \dfrac{6}{x+11} - \dfrac{2}{x-5}$.

4. $\dfrac{8x^2-13x-4}{x^3+x^2-2x} = \dfrac{8x^2-13x-4}{x(x+2)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-1} = \dfrac{A(x^2+x-2)+Bx(x-1)+Cx(x+2)}{x(x^2+x-2)}$

$= \dfrac{(A+B+C)x^2+(A-B+2C)x-2A}{x(x^2+x-2)}$.

This implies that $8x^2-13x-4 = (A+B+C)x^2+(A-B+2C)x-2A$.

$$\begin{cases} A+B+C=8 \\ A-B+2C=-13 \\ -2A=-4 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=3 \\ C=3 \end{cases}.$$

Therefore, $\dfrac{8x^2-13x-4}{x^3+x^2-2x} = \dfrac{2}{x} + \dfrac{3}{x+2} + \dfrac{3}{x-1}$.

5. $\dfrac{-x+15}{x^3+6x^2+5x} = \dfrac{-x+15}{x(x+5)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+5} + \dfrac{C}{x+1} =$

$$\frac{A(x^2+6x+5)+Bx(x+1)+Cx(x+5)}{x(x^2+6x+5)} = \frac{(A+B+C)x^2+(6A+B+5C)x+5A}{x(x^2+6x+5)}.$$

This implies that $-x + 15 = (A+B+C)x^2+(6A+B+5C)x+5A$.

$$\begin{cases} A+B+C=0 \\ 6A+B+5C=-1 \\ 5A=15 \end{cases} \Rightarrow \begin{cases} A=3 \\ B=1 \\ C=4 \end{cases}.$$

Therefore, $\dfrac{-x+15}{x^3+6x^2+5x} = \dfrac{3}{x} + \dfrac{1}{x+5} + \dfrac{4}{x+1}$.

6. $\dfrac{20x+2}{(x-2)(x+1)(x+5)} = \dfrac{A}{x-2} + \dfrac{B}{x+1} + \dfrac{C}{x+5} =$

$$\frac{A(x+1)(x+5)+B(x-2)(x+5)+C(x-2)(x+1)}{(x-2)(x+1)(x+5)} =$$

$$\frac{(A+B+C)x^2+(6A+3B-C)x+5A-10B-2C}{(x-2)(x+1)(x+5)}.$$

This implies that $20x + 2 = (A+B+C)x^2+(6A+3B-C)x+5A-10B-2C$.

$$\begin{cases} A+B+C=0 \\ 6A+3B-C=20 \\ 5A-10B-2C=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=\dfrac{3}{2} \\ C=-\dfrac{7}{2} \end{cases}.$$

Therefore, $\dfrac{20x+2}{(x-2)(x+1)(x+5)} = \dfrac{2}{x-2} + \dfrac{\frac{3}{2}}{x+1} + \dfrac{-\frac{7}{2}}{x+5}$.

7. $\dfrac{6x^2-4x+2}{x^3+x} = \dfrac{6x^2-4x+2}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} = \dfrac{A(x^2+1)+(Bx+C)x}{x(x^2+1)} = \dfrac{(A+B)x^2+Cx+A}{x(x^2+1)}$.

This implies that $6x^2-4x+2 = (A+B)x^2+Cx+A$.

$$\begin{cases} A+B=6 \\ C=-4 \\ A=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=4 \\ C=-4 \end{cases}.$$

Therefore, $\frac{6x^2-4x+2}{x^3+x} = \frac{2}{x} + \frac{4x-4}{x^2+1}$.

8. $\frac{x^2-4x-1}{(x-1)(x^2+x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+2} = \frac{A(x^2+x+2)+(Bx+C)(x-1)}{(x-1)(x^2+x+2)} =$

$\frac{(A+B)x^2+(A-B+C)x+2A-C}{(x-1)(x^2+x+2)}$.

This implies that $x^2-4x-1 = (A+B)x^2+(A-B+C)x+2A-C$.

$$\begin{cases} A+B=1 \\ A-B+C=-4 \\ 2A-C=-1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=2 \\ C=-1 \end{cases}.$$

Therefore, $\frac{x^2-4x-1}{(x-1)(x^2+x+2)} = \frac{-1}{x-1} + \frac{2x-1}{x^2+x+2}$.

9. $\frac{2x^2-4x}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)(x+2)}{(x+2)(x^2+4)} =$

$\frac{(A+B)x^2+(2B+C)x+4A+2C}{(x+2)(x^2+4)}$.

This implies that $2x^2-4x = (A+B)x^2+(2B+C)x+4A+2C$.

$$\begin{cases} A+B=2 \\ 2B+C=-4 \\ 4A+2C=0 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=0 \\ C=-4 \end{cases}.$$

Therefore, $\frac{x^2-4x-1}{(x-1)(x^2+x+2)} = \frac{2}{x+2} + \frac{-4}{x^2+4}$.

10. $\frac{4}{x(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} = \frac{A(x-1)(x^2+1)+Bx(x^2+1)+(Cx+D)(x-1)x}{x(x-1)(x^2+1)} =$

$\frac{(A+B+C)x^3+(-A+D-C)x^2+(A+B-D)x-A}{x(x-1)(x^2+1)}$.

This implies that 4 = $(A+B+C)x^3+(-A+D-C)x^2+(A+B-D)x-A$.

$$\begin{cases} A+B+C=0 \\ -A+D-C=0 \\ A+B-D=0 \\ -A=4 \end{cases} \Rightarrow \begin{cases} A=-4 \\ B=2 \\ C=2 \\ D=0 \end{cases}.$$

Therefore, $\dfrac{4}{x(x-1)(x^2+1)} = \dfrac{-4}{x} + \dfrac{2}{x-1} + \dfrac{2x}{x^2+1}$.

11. $\dfrac{2x+1}{x^3-x^2} = \dfrac{2x+1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} = \dfrac{Ax(x-1)+B(x-1)+Cx^3}{x^2(x-1)}$.

This implies that $2x + 1 = Ax(x-1)+B(x-1)+Cx^3$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = 0$ or $x = 1$.

Substitution $x = 0$ in $2x + 1 = Ax(x-1)+B(x-1)+Cx^3$ yields $B = -1$.

Substitution $x = 1$ in $2x + 1 = Ax(x-1)+B(x-1)+Cx^3$ yields $C = 3$.

Then $2x + 1 = Ax(x-1)-(x-1)+3x^3$.

Substitution $x = 2$ in $2x + 1 = Ax(x-1)-(x-1)+3x^3$ yields $A = -9$.

Therefore, $\dfrac{2x+1}{x^3-x^2} = \dfrac{-9}{x} + \dfrac{-1}{x^2} + \dfrac{3}{x-1}$.

12. $\dfrac{2x+12}{(x^2-2x)^2} = \dfrac{2x+12}{x^2(x-2)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} + \dfrac{D}{(x-2)^2} =$

$\dfrac{Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2}{x^2(x-2)^2}$.

This implies that $2x + 12 = Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = 0$ or $x = 2$.

Substitution $x = 0$ in $2x + 12 = Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2$ yields $B = 3$.

Substitution $x = 2$ in $2x + 12 = Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2$ yields $D = 4$.

Then $2x + 1 = Ax(x-2)^2+3(x-2)^2+Cx^2(x-2)+4x^2$.

Substitution $x = 1$ in $2x + 12 = Ax(x-2)^2+3(x-2)^2+Cx^2(x-2)+4x^2$ yields $A - C = 7$.

Substitution $x = 3$ in $2x + 12 = Ax(x-2)^2+3(x-2)^2+Cx^2(x-2)+4x^2$ yields $3A + 9C = -7$.

The solution of two equations in A and C is $A = \dfrac{14}{3}$ and $C = \dfrac{-7}{3}$.

Therefore, $\dfrac{2x+12}{(x^2-2x)^2} = \dfrac{\frac{14}{3}}{x} + \dfrac{3}{x^2} + \dfrac{\frac{-7}{3}}{x-2} + \dfrac{4}{(x-2)^2}$.

13. $\dfrac{36}{x(x-1)(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} =$

$\dfrac{A(x-1)(x+1)^2+Bx(x+1)^2+Cx(x^2-1)+Dx(x-1)}{x(x-1)(x+1)^2}$.

This implies that 36 = $A(x-1)(x+1)^2 + Bx(x+1)^2 + Cx(x^2-1) + Dx(x-1)$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = 0$, $x = -1$ or $x = 1$.

Substitution $x = 0$ in $36 = A(x-1)(x+1)^2 + Bx(x+1)^2 + Cx(x^2-1) + Dx(x-1)$ yields $A = -36$.

Substitution $x = -1$ in $36 = A(x-1)(x+1)^2 + Bx(x+1)^2 + Cx(x^2-1) + Dx(x-1)$ yields $D = 18$.

Substitution $x = 1$ in $36 = A(x-1)(x+1)^2 + Bx(x+1)^2 + Cx(x^2-1) + Dx(x-1)$ yields $B = 9$.

Then $36 = -36(x-1)(x+1)^2 + 9x(x+1)^2 + Cx(x^2-1) + 18x(x-1)$.

Substitution $x = -2$ in $36 = -36(x-1)(x+1)^2 + 9x(x+1)^2 + Cx(x^2-1) + 18x(x-1)$ yields $C = -9$.

Therefore, $\dfrac{36}{x(x-1)(x+1)^2} = \dfrac{-36}{x} + \dfrac{9}{x-1} + \dfrac{-9}{x+1} + \dfrac{18}{(x+1)^2}$.

14. $\dfrac{8}{(x^2-x)^2} = \dfrac{8}{x^2(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2} =$

$\dfrac{Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2}{x^2(x-1)^2}$.

This implies that $8 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = 0$ or $x = 1$.

Substitution $x = 0$ in $8 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$ yields $B = 8$.

Substitution $x = 1$ in $8 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$ yields $D = 8$.

Then $8 = Ax(x-1)^2 + 8(x-1)^2 + Cx^2(x-1) + 8x^2$.

Substitution $x = 2$ in $8 = Ax(x-1)^2 + 8(x-1)^2 + Cx^2(x-1) + 8x^2$ yields $A + 2C = -16$.

Substitution $x = -1$ in $8 = Ax(x-1)^2 + 8(x-1)^2 + Cx^2(x-1) + 8x^2$ yields $2A + C = 16$.

The solution of two equations in A and C is $A = 16$ and $C = -16$.

Therefore, $\dfrac{8}{(x^2-x)^2} = \dfrac{16}{x} + \dfrac{8}{x^2} + \dfrac{-16}{x-1} + \dfrac{8}{(x-1)^2}$.

15. $\dfrac{2x^2+50}{x^2(x+5)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+5} + \dfrac{D}{(x+5)^2} = \dfrac{Ax(x+5)^2 + B(x+5)^2 + Cx^2(x+5) + Dx^2}{x^2(x+5)^2}$.

This implies that $2x^2 + 50 = Ax(x+5)^2 + B(x+5)^2 + Cx^2(x+5) + Dx^2$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = 0$ or $x = -5$.

Substitution $x = 0$ in $2x^2 + 50 = Ax(x+5)^2 + B(x+5)^2 + Cx^2(x+5) + Dx^2$ yields $B = 2$.

Substitution $x = -5$ in $2x^2 + 50 = Ax(x+5)^2 + B(x+5)^2 + Cx^2(x+5) + Dx^2$ yields $D = 4$.

Then $2x^2 + 50 = Ax(x+5)^2 + 2(x+5)^2 + Cx^2(x+5) + 4x^2$.

Substitution $x = 1$ in $2x^2+50 = Ax(x+5)^2+2(x+5)^2+Cx^2(x+5)+4x^2$ yields $6A + C = –4$.

Substitution $x = –1$ in $2x^2+50 = Ax(x+5)^2+2(x+5)^2+Cx^2(x+5)+4x^2$ yields $–4A + C = 4$.

The solution of two equations in A and C is $A = \frac{-4}{5}$ and $C = \frac{4}{5}$.

Therefore, $\frac{2x^2+50}{x^2(x+5)^2} = \frac{\frac{-4}{5}}{x} + \frac{2}{x^2} + \frac{\frac{4}{5}}{x+5} + \frac{4}{(x+5)^2}$.

16. $\frac{4x+12}{(x-1)^2(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} =$

$\frac{A(x-1)^2(x+1)+B(x-1)^2+C(x-1)(x+1)^2+D(x+1)^2}{(x-1)^2(x+1)^2}$.

This implies that $4x + 12 = A(x-1)^2(x+1)+B(x-1)^2+C(x-1)(x+1)^2+D(x+1)^2$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = 1$ or $x = –1$.

Substitution $x = 1$ in $4x + 12 = A(x-1)^2(x+1)+B(x-1)^2+C(x-1)(x+1)^2+D(x+1)^2$ yields $D = 4$.

Substitution $x = –1$ in $4x + 12 = A(x-1)^2(x+1)+B(x-1)^2+C(x-1)(x+1)^2+D(x+1)^2$ yields $B = 2$.

Then $4x + 12 = A(x-1)^2(x+1)+2(x-1)^2+C(x-1)(x+1)^2+4(x+1)^2$.

Substitution $x = 0$ in $4x + 12 = A(x-1)^2(x+1)+2(x-1)^2+C(x-1)(x+1)^2+4(x+1)^2$ yields $A – C = 6$.

Substitution $x = 2$ in $4x + 12 = A(x-1)^2(x+1)+2(x-1)^2+C(x-1)(x+1)^2+4(x+1)^2$ yields $A + 3C = –6$.

The solution of two equations in A and C is $A = \frac{19}{3}$ and $C = \frac{1}{3}$.

Therefore, $\frac{4x+12}{(x-1)^2(x+1)^2} = \frac{\frac{19}{3}}{x+1} + \frac{2}{(x+1)^2} + \frac{\frac{1}{3}}{x-1} + \frac{4}{(x-1)^2}$.

17. $\frac{5}{(x^2+4)(x^2+1)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+1} = \frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x^2+4)(x^2+1)} =$

$\frac{(A+C)x^3+(B+D)x^2+(4A+C)x+4B+D}{(x^2+4)(x^2+1)}$.

This implies that 5 = $\left(A+C\right)x^3+\left(B+D\right)x^2+\left(4A+C\right)x+4B+D$.

$$\begin{cases} A+C=0 \\ B+D=0 \\ 4A+C=0 \\ 4B+D=5 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=\dfrac{5}{3} \\ C=0 \\ D=\dfrac{-5}{3} \end{cases}.$$

Therefore, $\dfrac{5}{\left(x^2+4\right)\left(x^2+1\right)} = \dfrac{\frac{5}{3}}{x^2+4} + \dfrac{-\frac{5}{3}}{x^2+1}$.

18. $\dfrac{1}{x^4+x^2} = \dfrac{1}{x^2\left(x^2+1\right)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1} = \dfrac{Ax\left(x^2+1\right)+B\left(x^2+1\right)+(Cx+D)x^2}{x^2\left(x^2+1\right)} =$

$\dfrac{\left(A+C\right)x^3+\left(B+D\right)x^2+Ax+B}{x^2\left(x^2+1\right)}$.

This implies that 5 = $\left(A+C\right)x^3+\left(B+D\right)x^2+Ax+B$.

$$\begin{cases} A+C=0 \\ B+D=0 \\ A=0 \\ B=1 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=1 \\ C=0 \\ D=-1 \end{cases}.$$

Therefore, $\dfrac{1}{x^4+x^2} = \dfrac{1}{x^2} + \dfrac{-1}{x^2+1}$.

19. $\dfrac{8x+9}{(x+1)^3(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{D}{x+2} =$

$\dfrac{A\left(x+1\right)^2\left(x+2\right)+B\left(x+1\right)\left(x+2\right)+C\left(x+2\right)+D\left(x+1\right)^3}{\left(x+1\right)^3\left(x+2\right)}$.

This implies that 8x + 9 = $A\left(x+1\right)^2\left(x+2\right)+B\left(x+1\right)\left(x+2\right)+C\left(x+2\right)+D\left(x+1\right)^3$.

The last equation is true for all $x\in\mathbb{R}$, even for x = –2 or x = –1.

Substitution x = –2 in 8x + 9 = $A\left(x+1\right)^2\left(x+2\right)+B\left(x+1\right)\left(x+2\right)+C\left(x+2\right)+D\left(x+1\right)^3$ yields D = 7.

Substitution x = –1 in 8x + 9 = $A\left(x+1\right)^2\left(x+2\right)+B\left(x+1\right)\left(x+2\right)+C\left(x+2\right)+D\left(x+1\right)^3$ yields C = 1.

Then 8x + 9 = $A\left(x+1\right)^2\left(x+2\right)+B\left(x+1\right)\left(x+2\right)+\left(x+2\right)+7\left(x+1\right)^3$.

Substitution $x = 0$ in $8x + 9 = A(x+1)^2(x+2)+B(x+1)(x+2)+(x+2)+7(x+1)^3$ yields $A + B = 0$.

Substitution $x = 1$ in $8x + 9 = A(x+1)^2(x+2)+B(x+1)(x+2)+(x+2)+7(x+1)^3$ yields $2A + B = -7$.

The solution of two equations in A and B is $A = -7$ and $B = 7$.

Therefore, $\dfrac{8x+9}{(x+1)^3(x+2)} = \dfrac{-7}{x+1} + \dfrac{7}{(x+1)^2} + \dfrac{1}{(x+1)^3} + \dfrac{7}{x+2}$.

20. $\dfrac{x^3+4x^2+x+2}{(x+1)(x-1)} = (x+4) + \dfrac{2x+7}{(x+1)(x-1)} = (x+4) + \dfrac{A}{x+1} + \dfrac{B}{x-1} = (x+4) + \dfrac{A(x-1)+B(x+1)}{(x+1)(x-1)} = (x+4) + \dfrac{(A+B)x-A+B}{(x+1)(x-1)}$.

This implies that $2x + 7 = (A+B)x-A+B$.

$$\begin{cases} A+B=2 \\ -A+B=7 \end{cases} \Rightarrow \begin{cases} A=-\dfrac{1}{2} \\ B=\dfrac{9}{2} \end{cases}.$$

Therefore, $\dfrac{x^3+4x^2+x+2}{(x+1)(x-1)} = (x+4) + \dfrac{-\frac{1}{2}}{x+1} + \dfrac{\frac{9}{2}}{x-1}$.

21. $\dfrac{2x^2+11x-10}{x(x-5)} = 2 + \dfrac{21x-10}{x(x-5)} = 2 + \dfrac{A}{x} + \dfrac{B}{x-5} = 2 + \dfrac{A(x-5)+Bx}{x(x-5)}$.

This implies that $21x - 10 = A(x-5)+Bx$.

$$\begin{cases} A+B=11 \\ -5A=10 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=13 \end{cases}.$$

Therefore, $\dfrac{2x^2+11x-10}{x(x-5)} = 2 + \dfrac{-2}{x} + \dfrac{13}{x-5}$.

22. $\dfrac{x-4}{x(x+2)} = \dfrac{A}{x} + \dfrac{B}{x+2} = \dfrac{(A+B)x+2A}{x(x+2)}$.

This implies that $x - 4 = (A + B)x + 2A$.

$$\begin{cases} A+B=1 \\ 2A=-4 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=3 \end{cases}.$$

Therefore, $\dfrac{x-4}{x(x+2)} = \dfrac{-2}{x} + \dfrac{3}{x+2}$.

$$\int \frac{x-4}{x(x+2)}\,dx = \int\left(\frac{-2}{x}+\frac{3}{x+2}\right)dx = -2\ln|x| + 3\ln|x+2| + C = \ln\frac{|x+2|^3}{x^2} + C.$$

23. $\frac{12x-6}{x^2+x-6} = \frac{12x-6}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{(A+B)x-2A+3B}{(x+3)(x-2)}$.

This implies that $12x - 6 = (A + B)x - 2A + 3B$.

$$\begin{cases} A+B=12 \\ -2A+3B=-6 \end{cases} \Rightarrow \begin{cases} A=\frac{42}{5} \\ B=\frac{18}{5} \end{cases}.$$

Therefore, $\frac{12x-6}{x^2+x-6} = \frac{\frac{42}{5}}{x+3} + \frac{\frac{18}{5}}{x-2}$.

$$\int \frac{12x-6}{x^2+x-6}\,dx = \frac{6}{5}\int\left(\frac{7}{x+3}+\frac{3}{x-2}\right)dx = \frac{6}{5}\left(7\ln|x+3|+3\ln|x-2|\right)+C =$$

$$\frac{6}{5}\ln\left(|x+3|^7\cdot|x-2|^3\right)+C.$$

24. $\frac{x^2+3}{x^3+x^2-2x} = \frac{x^2+3}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{Ñ}{x-1} = \frac{(A+B+C)x^2+(A-B+2C)x-2A}{x(x+2)(x-1)}$.

This implies that $x^2+3 = (A+B+C)x^2+(A-B+2C)x-2A$.

$$\begin{cases} A+B+C=1 \\ A-B+2C=0 \\ -2A=3 \end{cases} \Rightarrow \begin{cases} A=-\frac{3}{2} \\ B=-\frac{17}{6} \\ C=-\frac{2}{3} \end{cases}.$$

Therefore, $\frac{x^2+3}{x^3+x^2-2x} = \frac{-\frac{3}{2}}{x} + \frac{-\frac{17}{6}}{x+2} + \frac{-\frac{2}{3}}{x-1}$.

$$\int \frac{x^2+3}{x^3+x^2-2x}\,dx = \int\left(-\frac{3}{2}\cdot\frac{1}{x}-\frac{17}{6}\cdot\frac{1}{x+2}-\frac{2}{3}\cdot\frac{1}{x-1}\right)dx =$$

$$-\frac{3}{2}\ln|x|-\frac{17}{6}\ln|x+2|-\frac{2}{3}\ln|x-1|+C.$$

25. $\frac{100}{x^3-x^2-2x} = \frac{100}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{Ñ}{x+1} =$

$\frac{(A+B+C)x^2+(-A+B-2C)x-2A}{x(x-2)(x+1)}$.

This implies that $100 = (A+B+C)x^2+(-A+B-2C)x-2A$.

$$\begin{cases} A+B+C=0 \\ -A+B-2C=0 \\ -2A=100 \end{cases} \Rightarrow \begin{cases} A=-50 \\ B=\dfrac{50}{3} \\ C=\dfrac{100}{3} \end{cases}.$$

Therefore, $\dfrac{100}{x^3-x^2-2x} = \dfrac{-50}{x} + \dfrac{\frac{50}{3}}{x-2} + \dfrac{\frac{100}{3}}{x+1}$.

$\displaystyle\int \frac{100}{x^3-x^2-2x}\,dx = \frac{50}{3}\int\left(\frac{-3}{x}+\frac{1}{x-2}+\frac{2}{x+1}\right)dx = \frac{50}{3}\left(-3\ln|x|+\ln|x-2|+2\ln|x+1|\right)+C =$
$\dfrac{50}{3}\ln\left(|x-2|\cdot|x+1|^2\cdot|x|^{-3}\right)+C$.

26. $\dfrac{2}{x^3-2x^2+x} = \dfrac{2}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2} = \dfrac{(A+B)x^2+(-2A+B+C)x+A}{x(x-1)^2}$

This implies that 2 = $(A+B)x^2+(-2A+B+C)x+A$.

$$\begin{cases} A+B=0 \\ -2A+B+C=0 \\ A=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=-2 \\ C=6 \end{cases}.$$

Therefore, $\dfrac{2}{x^3-2x^2+x} = \dfrac{2}{x} + \dfrac{-2}{x-1} + \dfrac{6}{(x-1)^2}$.

$\displaystyle\int \frac{2}{x^3-2x^2+x}\,dx = \int\left(\frac{2}{x}+\frac{-2}{x-1}+\frac{6}{(x-1)^2}\right)dx$ = 2ln|x| – 2ln|x –1| + $\dfrac{-6}{x-1}$ + C = $\ln\dfrac{x^2}{(x-1)^2}$| +
$\dfrac{-6}{x-1}$ + C.

27. $\dfrac{4x}{(x-1)(x+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} = \dfrac{(A+B+C)x^2+(2A+C)x+A-B-C}{(x-1)(x+1)^2}$

This implies that 4x = $(A+B+C)x^2+(2A+C)x+A-B-C$.

$$\begin{cases} A+B=0 \\ 2A+C=4 \\ A-B-C=0 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=-3 \end{cases}.$$

Therefore, $\dfrac{4x}{(x-1)(x+1)^2} = \dfrac{1}{x-1} + \dfrac{-1}{x+1} + \dfrac{-3}{(x+1)^2}$.

$\displaystyle\int \frac{4x}{(x-1)(x+1)^2}\,dx = \int\left(\frac{1}{x-1}+\frac{-1}{x+1}+\frac{-3}{(x+1)^2}\right)dx = \ln|x-1|-\ln|x+1|+\frac{3}{x+1}+C =$
$\ln\dfrac{|x-1|}{|x+1|}+\dfrac{3}{x+1}+C$.

28. $\dfrac{2}{x^2(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2} = \dfrac{Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2}{x^2(x-1)^2}$.

This implies that 2 = $Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = 0$ or $x = 1$.

Substitution $x = 0$ in 2 = $Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$ yields $B = 2$.

Substitution $x = 1$ in 2 = $Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$ yields $D = 2$.

Then 2 = $Ax(x-1)^2 + 2(x-1)^2 + Cx^2(x-1) + 2x^2$.

Substitution $x = 2$ in 2 = $Ax(x-1)^2 + 2(x-1)^2 + Cx^2(x-1) + 2x^2$ yields $A + 2C = -4$.

Substitution $x = -1$ in 2 = $Ax(x-1)^2 + 2(x-1)^2 + Cx^2(x-1) + 2x^2$ yields $-2A - C = 1$.

The solution of two equations in A and C is $A = \dfrac{2}{3}$ and $C = \dfrac{-7}{3}$.

Therefore, $\dfrac{2}{x^2(x-1)^2} = \dfrac{2}{3x} + \dfrac{2}{x^2} + \dfrac{-7}{3(x-1)} + \dfrac{2}{(x-1)^2}$.

$$\int \frac{2}{x^2(x-1)^2}\,dx = \int\left(\frac{2}{3x} + \frac{2}{x^2} + \frac{-7}{3(x-1)} + \frac{2}{(x-1)^2}\right)dx = \frac{2}{3}\ln|x| - \frac{7}{3}\ln|x-1| - \frac{2}{x} - \frac{2}{x-1} + C =$$
$$\frac{1}{3}\ln\frac{x^2}{|x-1|^7} - \frac{2}{x} - \frac{2}{x-1} + C.$$

29. $\dfrac{5x+1}{x^2-2x+1} = \dfrac{5x+1}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} = \dfrac{A(x-1)+B}{(x-1)^2}$

This implies that $5x + 1 = A(x-1) + B$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = 1$.

Substitution $x = 1$ in $5x + 1 = A(x-1) + B$ yields $B = 6$.

Then $5x + 1 = A(x-1) + 6$.

Substitution $x = 0$ in $5x + 1 = A(x-1) + 6$ yields $A = 5$.

Therefore, $\dfrac{5x+1}{x^2-2x+1} = \dfrac{5}{x-1} + \dfrac{6}{(x-1)^2}$.

$$\int \frac{5x+1}{x^2-2x+1}\,dx = \int\left(\frac{5}{x-1} + \frac{6}{(x-1)^2}\right)dx = 5\ln|x-1| - \frac{6}{x-1} + C.$$

30. $\dfrac{2x+1}{x(x+2)^2(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2} + \dfrac{D}{x-1} + \dfrac{E}{(x-1)^2} =$

$$\frac{A(x+2)^2(x-1)^2 + Bx(x+2)(x-1)^2 + Cx(x-1)^2 + Dx(x+2)^2(x-1) + Ex(x+2)^2}{x(x+2)^2(x-1)^2}.$$

This implies that $2x + 1 =$
$A(x+2)^2(x-1)^2 + Bx(x+2)(x-1)^2 + Cx(x-1)^2 + Dx(x+2)^2(x-1) + Ex(x+2)^2$.
The last equation is true for all $x \in \mathbb{R}$, even for $x = -2$, $x = 0$ or $x = 1$.
Substitution $x = -2$ in $2x + 1 =$

$A(x+2)^2(x-1)^2 + Bx(x+2)(x-1)^2 + Cx(x-1)^2 + Dx(x+2)^2(x-1) + Ex(x+2)^2$ yields $C = \frac{1}{6}$.

Substitution $x = 0$ in $2x + 1 =$

$A(x+2)^2(x-1)^2 + Bx(x+2)(x-1)^2 + Cx(x-1)^2 + Dx(x+2)^2(x-1) + Ex(x+2)^2$ yields $A = \frac{1}{4}$.

Substitution $x = 1$ in $2x + 1 =$

$A(x+2)^2(x-1)^2 + Bx(x+2)(x-1)^2 + Cx(x-1)^2 + Dx(x+2)^2(x-1) + Ex(x+2)^2$ yields $E = \frac{1}{3}$.

Then $2x + 1 =$
$\frac{1}{4}(x+2)^2(x-1)^2 + Bx(x+2)(x-1)^2 + \frac{1}{6}x(x-1)^2 + Dx(x+2)^2(x-1) + \frac{1}{3}x(x+2)^2$.
Substitution $x = -1$ in $2x + 1 =$
$\frac{1}{4}(x+2)^2(x-1)^2 + Bx(x+2)(x-1)^2 + \frac{1}{6}x(x-1)^2 + Dx(x+2)^2(x-1) + \frac{1}{3}x(x+2)^2$ yields $4B - 2D = \frac{7}{3}$.

Substitution $x = 2$ in $2x + 1 =$
$\frac{1}{4}(x+2)^2(x-1)^2 + Bx(x+2)(x-1)^2 + \frac{1}{6}x(x-1)^2 + Dx(x+2)^2(x-1) + \frac{1}{3}x(x+2)^2$ yields $4B + 4D = \frac{1}{3}$.

The solution of two equations in B and D is $B = -1$ and $D = \frac{-1}{3}$.

Therefore, $\dfrac{2x+1}{x(x+2)^2(x-1)^2} = \dfrac{1}{4x} + \dfrac{-1}{x+2} + \dfrac{1}{6(x+2)^2} + \dfrac{1}{3(x-1)} + \dfrac{1}{3(x-1)^2}$.

$$\int \frac{2x+1}{x(x+2)^2(x-1)^2}\,dx = \int \left(\frac{1}{4x} + \frac{-1}{x+2} + \frac{1}{6(x+2)^2} + \frac{1}{3(x-1)} + \frac{1}{3(x-1)^2} \right) dx =$$
$$\frac{1}{4}\ln|x| - \ln|x+2| - \frac{1}{6(x+2)} + \frac{1}{3}\ln|x-1| - \frac{1}{3(x-1)} + C.$$

31. $\dfrac{9x^2 - x + 10}{x^3 + 2x} = \dfrac{9x^2 - x + 10}{x(x^2+2)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 2} = \dfrac{(A+B)x^2 + Cx + 2A}{x(x^2+2)}$.

This implies that $9x^2 - x + 10 = \left(A+B\right)x^2 + Cx + 2A$

$$\begin{cases} A+B=9 \\ C=-1 \\ -2A=10 \end{cases} \Rightarrow \begin{cases} A=-5 \\ B=14 \\ C=-1 \end{cases}.$$

Therefore, $\dfrac{9x^2 - x + 10}{x^3 + 2x} = \dfrac{-5}{x} + \dfrac{14x-1}{x^2+2}$.

$$\int \frac{9x^2 - x + 10}{x^3 + 2x}\,dx = \int\left(\frac{-5}{x} + \frac{14x-1}{x^2+2}\right)dx = \int\left(\frac{-5}{x} + \frac{14x}{x^2+2} - \frac{1}{x^2+2}\right)dx =$$

$$-5\ln|x| + 7\ln(x^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C.$$

32. $\dfrac{1}{x\left(x^2+1\right)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} = \dfrac{(A+B)x^2 + Cx + A}{x\left(x^2+1\right)}$.

This implies that $1 = (A+B)x^2 + Cx + A$

$$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=0 \end{cases}.$$

Therefore, $\dfrac{1}{x\left(x^2+1\right)} = \dfrac{1}{x} + \dfrac{-x}{x^2+1}$.

$$\int \frac{1}{x^3+x}\,dx = \int\left(\frac{1}{x} + \frac{-x}{x^2+1}\right)dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + C.$$

33. $\dfrac{5x^3 + 5x^2 + 7x + 20}{x^4 + 4x^2} = \dfrac{5x^3 + 5x^2 + 7x + 20}{x^2\left(x^2+4\right)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+4} =$

$$\frac{(A+C)x^3 + (B+D)x^2 + 4Ax + 4B}{x^2\left(x^2+4\right)}.$$

This implies that $5x^3 + 5x^2 + 7x + 20 = (A+C)x^3 + (B+D)x^2 + 4Ax + 4B$

$$\begin{cases} A+C=5 \\ B+D=5 \\ 4A=7 \\ 4B=20 \end{cases} \Rightarrow \begin{cases} A=\dfrac{7}{4} \\ B=5 \\ C=\dfrac{13}{4} \\ D=0 \end{cases}.$$

Therefore, $\dfrac{5x^3 + 5x^2 + 7x + 20}{x^4 + 4x^2} = \dfrac{7}{4x} + \dfrac{5}{x^2} + \dfrac{13x}{4(x^2+4)}$.

$$\int \frac{5x^3 + 5x^2 + 7x + 20}{x^4 + 4x^2}\,dx = \int\left(\frac{7}{4x} + \frac{5}{x^2} + \frac{13x}{4(x^2+4)}\right)dx = \frac{7}{4}\ln|x| - \frac{5}{x} + \frac{13}{8}\ln(x^2+4) + C.$$

34. $\dfrac{2x^2 + 11x - 10}{x\left(x-5\right)} = 2 + \dfrac{21x-10}{x\left(x-5\right)} = 2 + \dfrac{A}{x} + \dfrac{B}{x-5} = 2 + \dfrac{(A+B)x - 5A}{x\left(x-5\right)}$.

This implies that $21x - 10 = (A+B)x - 5A$.

$$\begin{cases} A+B=21 \\ -5A=10 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=23 \end{cases}.$$

Therefore, $\dfrac{2x^2+11x-10}{x(x-5)} = 2 + \dfrac{-2}{x} + \dfrac{23}{x-5}$.

$$\int \frac{2x^2+11x-10}{x(x-5)}\,dx = \int\left(2+\frac{-2}{x}+\frac{23}{x-5}\right)dx = 2x-2\ln|x|+23\ln|x-5|+C.$$

35. $\dfrac{x^2+4}{x^2-4} = 1 + \dfrac{8}{(x-2)(x+2)} = 1 + \dfrac{A}{x-2} + \dfrac{B}{x+2} = 1 + \dfrac{(A+B)x-2A+2B}{(x-2)(x+2)}$.

This implies that $8 = (A+B)x-2A+2B$.

$$\begin{cases} A+B=0 \\ -2A+2B=8 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=2 \end{cases}.$$

Therefore, $\dfrac{x^2+4}{x^2-4} = 1 + \dfrac{-2}{x-2} + \dfrac{2}{x+2}$.

$$\int \frac{x^2+4}{x^2-4}\,dx = \int\left(1+\frac{-2}{x-2}+\frac{2}{x+2}\right)dx = x-2\ln|x-2|+2\ln|x+2|+C = x+\ln\frac{(x+2)^2}{(x-2)^2}+C.$$

36. $\dfrac{1}{x^4-16} = \dfrac{1}{(x-2)(x+2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+4} =$

$$\frac{A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x^2-4)}{(x-2)(x+2)(x^2+4)}.$$

This implies that $1 = A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x^2-4)$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = -2$ or $x = 2$.

Substitution $x = -2$ in $1 = A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x^2-4)$ yields $B = \dfrac{-1}{32}$.

Substitution $x = 2$ in $1 = A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x^2-4)$ yields $A = \dfrac{1}{32}$.

Then $1 = \dfrac{1}{32}(x+2)(x^2+4)-\dfrac{1}{32}(x-2)(x^2+4)+(Cx+D)(x^2-4)$.

Substitution $x = 0$ in $1 = \dfrac{1}{32}(x+2)(x^2+4)-\dfrac{1}{32}(x-2)(x^2+4)+(Cx+D)(x^2-4)$ yields $D = \dfrac{1}{8}$.

Substitution $x = -1$ in $1 = \dfrac{1}{32}(x+2)(x^2+4)-\dfrac{1}{32}(x-2)(x^2+4)+(Cx+D)(x^2-4)$ yields $C = \dfrac{1}{4}$.

Therefore, $\dfrac{1}{x^4-16} = \dfrac{1}{32(x-2)} + \dfrac{-1}{32(x+2)} + \dfrac{\frac{1}{4}x+\frac{1}{8}}{x^2+4}$.

$$\int \frac{1}{x^4-16}\,dx = \int\left(\frac{1}{32(x-2)}+\frac{-1}{32(x+2)}+\frac{2x+1}{8(x^2+4)}\right)dx =$$

$$\frac{1}{32}\ln|x-2|-\frac{1}{32}\ln|x+2|+\frac{1}{8}\ln(x^2+4)+\frac{1}{16}\tan^{-1}\frac{x}{2}+C =$$

$$\frac{1}{32}\ln\frac{|x-2|(x^2+4)^4}{|x+2|}+\frac{1}{16}\tan^{-1}\frac{x}{2}+C.$$

37. $\frac{x^3+4x^2+x+2}{(x+1)(x-1)} = x + 4 + \frac{2x+6}{(x+1)(x-1)} = x + 4 + \frac{A}{x+1} + \frac{B}{x-1} = x + 4 + \frac{(A+B)x-A+B}{(x+1)(x-1)}$

This implies that $2x + 6 = (A+B)x-A+B$.

$$\begin{cases} A+B=2 \\ -A+B=6 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=4 \end{cases}.$$

Therefore, $\frac{x^3+4x^2+x+2}{(x+1)(x-1)} = x + 4 + \frac{-2}{x+1} + \frac{4}{x-1}$.

$$\int \frac{x^3+4x^2+x+2}{(x+1)(x-1)}\,dx = \int\left(x+4-\frac{2}{x+1}+\frac{4}{x-1}\right)dx = \frac{x^2}{2}+4x-2\ln|x+1|+4\ln|x-1|+C =$$

$$\frac{x^2}{2}+4x+\ln\frac{(x-1)^4}{(x+1)^2}+C.$$

38. Let $\theta = e^x$ then $\int \frac{e^x}{e^{2x}-1}\,dx = \int \frac{1}{\theta^2-1}\,d\theta$.

$$\frac{1}{\theta^2-1} = \frac{1}{(\theta-1)(\theta+1)} = \frac{A}{\theta-1} + \frac{B}{\theta+1} = \frac{(A+B)\theta+A-B}{(\theta-1)(\theta+1)}.$$

This implies that $1 = (A+B)\theta+A-B$.

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}.$$

Therefore, $\frac{1}{\theta^2-1} = \frac{1}{2(\theta-1)} - \frac{1}{2(\theta+1)}$.

$$\int \frac{1}{\theta^2-1}\,d\theta = \int\left(\frac{1}{2(\theta-1)}-\frac{1}{2(\theta+1)}\right)d\theta = \frac{1}{2}\left(\ln|\theta-1|-\ln|\theta+1|\right)+C = \frac{1}{2}\ln\frac{|e^x-1|}{|e^x+1|}+C.$$

39. Let $\theta = \sqrt{x}$ then $\int \frac{4}{\sqrt{x}(x-1)}\,dx = \int \frac{8}{\theta^2-1}\,d\theta = 8\int\left(\frac{1}{2(\theta-1)}-\frac{1}{2(\theta+1)}\right)d\theta =$

$$4\left(\ln|\theta-1|-\ln|\theta+1|\right)+C = 4\ln\frac{|\theta-1|}{|\theta+1|}+C = 4\ln\frac{|\sqrt{x}-1|}{|\sqrt{x}+1|}+C.$$

40. a) If $u=\tan\frac{\theta}{2}$ then $\theta = 2\tan^{-1}u$.

$$\sin\theta = \sin(2\tan^{-1}u) = 2\sin(\tan^{-1}u)\cos(\tan^{-1}u) = 2\frac{u}{\sqrt{1+u^2}}\cdot\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}.$$

$\cos\theta = \cos(2\tan^{-1}u) = 1 - 2\sin^2(\tan^{-1}u) = 1 - 2\left(\dfrac{u}{\sqrt{1+u^2}}\right)^2 = 1 - \dfrac{2u^2}{1+u^2} = \dfrac{1-u^2}{1+u^2}$.

$d\theta = d(2\tan^{-1}u) = \dfrac{2\,du}{1+u^2}$.

b) $\displaystyle\int \frac{\sin\theta}{\sin\theta-\cos\theta}\,d\theta = \int \frac{\frac{2u}{1+u^2}}{\frac{2u}{1+u^2}-\frac{1-u^2}{1+u^2}}\cdot\frac{2\,du}{1+u^2} = \int \frac{4u}{\left(u^2+2u-1\right)\left(1+u^2\right)}du$

$\dfrac{4u}{\left(u^2+2u-1\right)\left(1+u^2\right)} = \dfrac{Au+B}{u^2+2u-1} + \dfrac{Cu+D}{1+u^2} =$

$\dfrac{(A+C)u^3+(B+2C+D)u^2+(A-C+2D)u+B-D}{\left(u^2+2u-1\right)\left(1+u^2\right)}$.

This implies that $4u = (A+C)u^3+(B+2C+D)u^2+(A-C+2D)u+B-D$.

$\begin{cases} A+C=0 \\ B+2C+D=0 \\ A-C+2D=4 \\ B-D=0 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \\ C=-1 \\ D=1 \end{cases}$.

Therefore, $\dfrac{4u}{\left(u^2+2u-1\right)\left(1+u^2\right)} = \dfrac{u+1}{u^2+2u-1} + \dfrac{-u+1}{1+u^2}$.

$\displaystyle\int \frac{4u}{\left(u^2+2u-1\right)\left(1+u^2\right)}du = \int\left(\frac{u+1}{u^2+2u-1}+\frac{-u+1}{1+u^2}\right)du =$

$\dfrac{1}{2}\ln|u^2+2u-1| - \dfrac{1}{2}\ln(u^2+1) + \tan^{-1}u + C = \dfrac{1}{2}\ln\dfrac{|u^2+2u-1|}{u^2+1} + \tan^{-1}u + C =$

$\dfrac{1}{2}\ln\dfrac{|\tan^2\frac{\theta}{2}+2\tan\frac{\theta}{2}-1|}{\tan^2\frac{\theta}{2}+1} + \dfrac{\theta}{2} + C = \dfrac{1}{2}\ln|\sin\theta-\cos\theta| + \dfrac{\theta}{2} + C$.

41. $\dfrac{x^4(1-x)^4}{1+x^2} = \dfrac{x^8-4x^7+6x^6-4x^5+x^4}{1+x^2} = x^6-4x^5+5x^4-4x^2+4 - \dfrac{4}{1+x^2}$.

$\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx =$

$\left.\left(\dfrac{x^7}{7}-\dfrac{4x^6}{6}+x^5-\dfrac{4x^3}{3}+4x-4\tan^{-1}x\right)\right|_0^1 = \dfrac{1}{7}-\dfrac{4}{6}+1-\dfrac{4}{3}+4-4\tan^{-1}1 = \dfrac{22}{7}-\pi$.

***42.** $\dfrac{1}{1+x^4} = \dfrac{1}{\left(1+x^2\right)^2-2x^2} = \dfrac{1}{\left(1+x^2+\sqrt{2}x\right)\left(1+x^2-\sqrt{2}x\right)} = \dfrac{Ax+B}{1+x^2+\sqrt{2}x} + \dfrac{Cx+D}{1+x^2-\sqrt{2}x} =$

$\dfrac{(Ax+B)\left(1+x^2-\sqrt{2}x\right)+(Cx+D)\left(1+x^2+\sqrt{2}x\right)}{\left(1+x^2-\sqrt{2}x\right)\left(1+x^2+\sqrt{2}x\right)}$.

This implies that 1 = $(Ax+B)\left(1+x^2-\sqrt{2}x\right)+(Cx+D)\left(1+x^2+\sqrt{2}x\right)$.

$$\begin{cases} A+C=0 \\ -\sqrt{2}A+B+\sqrt{2}C+D=0 \\ A-\sqrt{2}B+C+\sqrt{2}D=0 \\ B+D=1 \end{cases} \Rightarrow \begin{cases} A=\dfrac{1}{2\sqrt{2}} \\ B=\dfrac{1}{2} \\ C=-\dfrac{1}{2\sqrt{2}} \\ D=\dfrac{1}{2} \end{cases}.$$

Therefore, $\dfrac{1}{1+x^4} = \dfrac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{1+x^2+\sqrt{2}x} + \dfrac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{1+x^2-\sqrt{2}x}$.

$$\int\frac{1}{1+x^4}dx = \frac{1}{2\sqrt{2}}\int\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}-\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)dx = \frac{1}{2\sqrt{2}}\int\frac{x+\frac{\sqrt{2}}{2}}{x^2+\sqrt{2}x+1}dx \;-+$$

$$\frac{1}{4}\int\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx - \frac{1}{2\sqrt{2}}\int\frac{x-\frac{\sqrt{2}}{2}}{x^2-\sqrt{2}x+1}dx + \frac{1}{4}\int\frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}dx = \frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}$$

$$+ \frac{1}{2\sqrt{2}}\left(\tan^{-1}(x\sqrt{2}+1)+\tan^{-1}(x\sqrt{2}-1)\right) + C.$$

3. Integration by Parts

1. Let $u = x$ and $dv = \sin 2x\,dx$ then $du = dx$ and $v = -\frac{\cos 2x}{2}$.

$$\int x\sin 2x\,dx = -\frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx = -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + C.$$

2. Let $u = y$ and $dv = \cos 3y\,dy$ then $du = dy$ and $v = \frac{\sin 3y}{3}$.

$$\int 5y\cos 3y\,dy = \frac{5}{3}y\sin 3y - \frac{5}{3}\int \sin 3y\,dy = \frac{5}{3}y\sin 3y + \frac{5}{9}\cos 3y + C.$$

3. Let $u = t$ and $dv = e^t\,dt$ then $du = dt$ and $v = e^t$.

$$\int_0^1 2t\,e^t\,dt = 2te^t\Big|_0^1 - 2\int_0^1 e^t\,dt = 2te^t\Big|_0^1 - 2e^t\Big|_0^1 = 2e - 2e + 2 = 2.$$

4. Let $u = x$ and $dv = e^{5x}\,dx$ then $du = dx$ and $v = \frac{e^{5x}}{5}$.

$$\int 4xe^{5x}\,dx = \frac{4}{5}xe^{5x} - \frac{4}{5}\int e^{5x}\,dx = \frac{4}{5}xe^{5x} - \frac{4}{25}e^{5x} + C.$$

5. Let $u = x$ and $dv = \left(e^{4x}+e^{-4x}\right)dx$ then $du = dx$ and $v = \frac{e^{4x}-e^{-4x}}{4}$.

$$\int 2x\left(e^{4x}+e^{-4x}\right)dx = \frac{2x\left(e^{4x}-e^{-4x}\right)}{4} - \frac{1}{2}\int\left(e^{4x}-e^{-4x}\right)dx = \frac{x\left(e^{4x}-e^{-4x}\right)}{2} - \frac{e^{4x}+e^{-4x}}{8} + C.$$

6. Let $u = x$ and $dv = \left(e^{3x}-e^{-3x}\right)dx$ then $du = dx$ and $v = \frac{e^{3x}+e^{-3x}}{3}$.

$$\int 8x\left(e^{3x}-e^{-3x}\right)dx = \frac{8x(e^{3x}+e^{-3x})}{3} - \frac{8}{3}\int\left(e^{3x}+e^{-3x}\right)dx = \frac{8x(e^{3x}+e^{-3x})}{3} - \frac{8}{9}\left(e^{3x}-e^{-3x}\right) + C.$$

7. Let $u = e^{2x}$ and $dv = \cos 4x\,dx$ then $du = 2e^{2x}dx$ and $v = \frac{\sin 4x}{4}$.

$$\int_0^{\pi/2} e^{2x}\cos 4x\,dx = \frac{e^{2x}\sin 4x}{4}\Bigg|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} e^{2x}\sin 4x\,dx = \frac{e^{2x}\sin 4x}{4}\Bigg|_0^{\pi/2} + \frac{e^{2x}\cos 4x}{8}\Bigg|_0^{\pi/2} -$$
$$\frac{1}{4}\int_0^{\pi/2} e^{2x}\cos 4x\,dx.$$

Therefore, $\int_0^{\pi/2} e^{2x}\cos 4x\,dx = \frac{4}{5}\left(\frac{e^{2x}\sin 4x}{4}\Bigg|_0^{\pi/2} + \frac{e^{2x}\cos 4x}{8}\Bigg|_0^{\pi/2}\right) = \frac{4}{5}\left(\frac{e^{\pi}}{8}-\frac{1}{8}\right) = \frac{e^{\pi}}{10}-\frac{1}{10}$.

8. Let $u = e^{3x}$ and $dv = \sin 5x\,dx$ then $du = 3e^{3x}dx$ and $v = -\frac{\cos 5x}{5}$.

$$\int e^{3x}\sin 5x\,dx = -\frac{e^{3x}\cos 5x}{5} + \frac{3}{5}\int e^{3x}\cos 5x\,dx = -\frac{e^{3x}\cos 5x}{5} + \frac{3e^{3x}\sin 5x}{25} -$$
$$\frac{9}{25}\int e^{3x}\sin 5x\,dx.$$

Therefore, $\int e^{3x}\sin 5x\,dx = \frac{25}{34}\left(-\frac{e^{3x}\cos 5x}{5}+\frac{3e^{3x}\sin 5x}{25}\right)+C = \frac{e^{3x}}{34}\left(-5\cos 5x+3\sin 5x\right)+C$.

9. Let $u = \ln(4x)$ and $dv = dx$ then $du = \dfrac{dx}{x}$ and $v = x$.

$\int \ln(4x)\,dx = x\ln(4x) - \int dx = x\ln(4x) - x + C.$

10. Let $u = x$ and $dv = 2^{-x}dx$ then $du = dx$ and $v = -2^{-x}\ln 2$.

$$\int_0^1 x\,2^{-x}\,dx = -2^{-x}\ln 2\cdot x\Big|_0^1 + \int_0^1 2^{-x}\ln 2\,dx = -2^{-x}\ln 2\cdot x\Big|_0^1 - 2^{-x}\Big|_0^1 = -\frac{\ln 2}{2} + \frac{1}{2}.$$

11. Let $u = \log_4 x$ and $dv = dx$ then $du = \dfrac{dx}{x\ln 4}$ and $v = x$.

$$\int \log_4 x\,dx = x\log_4 x - \int \frac{x}{x\ln 4}dx = x\log_4 x - \frac{x}{\ln 4} + C.$$

12. Let $u = \ln(2x)$ and $dv = xdx$ then $du = \dfrac{dx}{x}$ and $v = \dfrac{x^2}{2}$.

$$\int_1^{e/2} x\ln(2x)\,dx = \frac{x^2}{2}\ln(2x)\Big|_1^{e/2} - \int_1^{e/2}\frac{x}{2}\,dx = \frac{x^2}{2}\ln(2x)\Big|_1^{e/2} - \frac{x^2}{4}\Big|_1^{e/2} = \frac{e^2}{8} - \frac{\ln 2}{2} - \frac{e^2}{16} + \frac{1}{4} = \frac{e^2}{16} - \frac{\ln 2}{2} + \frac{1}{4}.$$

13. Let $u = \ln(4x)$ and $dv = x^2\,dx$ then $du = \dfrac{dx}{x}$ and $v = \dfrac{x^3}{3}$.

$$\int x^2\ln(4x)\,dx = \frac{x^3}{3}\ln(4x) - \int\frac{x^2}{3}\,dx = \frac{x^3}{3}\ln(4x) - \frac{x^3}{9} + C.$$

14. Let $u = \sin^{-1}(2x)$ and $dv = dx$ then $du = \dfrac{2dx}{\sqrt{1-4x^2}}$ and $v = x$.

$$\int \sin^{-1}(2x)\,dx = x\sin^{-1}(2x) - \int\frac{2xdx}{\sqrt{1-4x^2}}\,dx = x\sin^{-1}(2x) + \sqrt{1-4x^2} + C.$$

15. Let $u = \tan^{-1}(4y)$ and $dv = dy$ then $du = \dfrac{4dy}{1+16y^2}$ and $v = y$.

$$\int \tan^{-1}(4y)\,dy = y\tan^{-1}(4y) - \int\frac{4ydy}{1+16y^2}\,dy = y\tan^{-1}(4y) - \frac{1}{8}\ln(1+16y^2) + C.$$

16. Let $u = r^2$ and $dv = \cos r dr$ then $du = 2rdr$ and $v = \sin r$.

$$\int_0^{\pi/2} r^2\cos r\,dr = r^2\sin r\Big|_0^{\pi/2} - 2\int_0^{\pi/2} r\sin r\,dr = r^2\sin r\Big|_0^{\pi/2} + 2r\cos r\Big|_0^{\pi/2} - 2\int_0^{\pi/2}\cos r\,dr =$$

$$r^2\sin r\Big|_0^{\pi/2} + 2r\cos r\Big|_0^{\pi/2} - \sin r\Big|_0^{\pi/2} = \pi.$$

17. Let $u = \sec x$ and $dv = \sec^2 x\,dx$ then $du = \sin x\sec^2 x dx$ and $v = \tan x$.

$\int \sec^3 x\,dx = \tan x\sec x - \int \sin^2 x\sec^3 x\,dx = \tan x\sec x - \int\left(\sec^3 x - \sec x\right)dx = \tan x\sec x - \int \sec^3 x\,dx + \ln|\sec x + \tan x| + C.$

Therefore, $\int \sec^3 x\,dx = \dfrac{1}{2}\tan x\sec x + \dfrac{1}{2}\ln|\sec x + \tan x| + C.$

18. Let $u = \sec^{-1} x$ and $dv = xdx$ then $du = \dfrac{\sec^{-2} x}{\sqrt{1-x^2}}dx$ and $v = \dfrac{x^2}{2}$.

$\int x\sec^{-1}x\,dx = \frac{x^2}{2}\sec^{-1}x - \int\frac{x^2\sec^{-2}x}{2\sqrt{1-x^2}}dx = \frac{x^2}{2}\sec^{-1}x + \frac{1}{2}\int\left(\sqrt{1-x^2}\sec^{-2}x - \frac{\sec^{-2}x}{\sqrt{1-x^2}}\right)dx$

$= \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\sec^{-1}x + \frac{1}{2}\int\sqrt{1-x^2}\sec^{-2}x\,dx.$

Let $u = 1-x^2$ and $dv = \frac{\sec^{-2}x}{\sqrt{1-x^2}}dx$ then $du = -2x$ and $v = -\sec^{-1}x$.

$\frac{1}{2}\int\sqrt{1-x^2}\sec^{-2}x\,dx = -\frac{(1-x^2)}{2}\sec^{-1}x - \int x\sec^{-1}x\,dx + C.$

Therefore, $\int x\sec^{-1}x\,dx = \frac{x^2}{4}\sec^{-1}x - \frac{1}{4}\sec^{-1}x - \frac{(1-x^2)}{4}\sec^{-1}x + C.$

19. Let $u = \sin^{-1}x^2$ and $dv = xdx$ then $du = \frac{2x}{\sqrt{1-x^4}}dx$ and $v = \frac{x^2}{2}$.

$\int x\sin^{-1}x^2\,dx = \frac{x^2}{2}\sin^{-1}x^2 - \int\frac{x^3}{\sqrt{1-x^4}}dx = \frac{x^2}{2}\sin^{-1}x^2 - \frac{1}{2}\sqrt{1-x^4} + C.$

20. Let $u = x^2+1$ and $dv = \cos x dx$ then $du = 2xdx$ and $v = \sin x$.

$\int(x^2+1)\cos x\,dx = (x^2+1)\sin x - 2\int x\sin x\,dx = (x^2+1)\sin x + 2x\cos x - 2\int\cos x\,dx =$
$(x^2+1)\sin x + 2x\cos x - 2\sin x + C = (x^2-1)\sin x + 2x\cos x + C.$

21. Let $u = x^2+x+2$ and $dv = e^x dx$ then $du = (2x+1)dx$ and $v = e^x$.

$\int(x^2+x+2)e^x\,dx = (x^2+x+2)e^x - \int(2x+1)e^x\,dx = (x^2+x+2)e^x - (2x+1)e^x + 2\int e^x\,dx$
$= (x^2+x+2)e^x - (2x+1)e^x + 2e^x + C = (x^2-x+3)e^x + C.$

22. Let $u = x^3$ and $dv = \sin x dx$ then $du = 3x^2dx$ and $v = -\cos x$.

$\int x^3\sin x\,dx = -x^3\cos x + 3\int x^2\cos x\,dx = -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx = -x^3\cos x +$
$3x^2\sin x + 6x\cos x - 6\int\cos x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.$

23. Let $u = \tan^{-1}x$ and $dv = xdx$ then $du = \frac{dx}{1+x^2}$ and $v = \frac{x^2}{2}$.

$\int x\tan^{-1}x\,dx = \frac{x^2\tan^{-1}x}{2} - \int\frac{x^2}{2(1+x^2)}dx = \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx = \frac{x^2\tan^{-1}x}{2} - \frac{x}{2}$
$+ \frac{\tan^{-1}x}{2} + C.$

24. Let $u = \sin^{-1}(2x)$ and $dv = xdx$ then $du = \frac{2dx}{\sqrt{1-4x^2}}$ and $v = \frac{x^2}{2}$.

$\int x\sin^{-1}(2x)\,dx = \frac{x^2\sin^{-1}(2x)}{2} - \int\frac{x^2}{\sqrt{1-4x^2}}dx = \frac{x^2\sin^{-1}(2x)}{2} +$
$\frac{1}{4}\int\left(\sqrt{1-4x^2} - \frac{1}{\sqrt{1-4x^2}}\right)dx = \frac{x^2\sin^{-1}(2x)}{2} + \frac{x}{8}\sqrt{1-4x^2} + \frac{1}{16}\sin^{-1}(2x) - \frac{1}{8}\sin^{-1}(2x) + C =$
$\frac{x^2\sin^{-1}(2x)}{2} + \frac{x}{8}\sqrt{1-4x^2} + \frac{1}{8}\sin^{-1}(2x) + C.$

25. The volume of the solid is $\int_0^{\pi}\sin x\,dx = \cos x\Big|_0^{\pi} = 1 + 1 = 2.$

26. The volume of the solid is $\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx = \sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{\pi}$ = 1 + 0 – 0 + 1 = 2.

27. The length of the curve is $\int_0^1 \left(2x^2\right)' dx = 2x^2\Big|_0^1 = 2.$

28. Let $u = \ln x$ then $x = e^u$ and $dx = e^u du$.

$\int \sin(\ln x)\, dx = \int \sin u\, e^u\, du$ and the integral can be evaluated using integration by parts.

Let $w = e^u$ and $dv = \sin u du$ then $dw = e^u du$ and $v = -\cos u$.

$\int \sin u\, e^u\, du = -e^u \cos u + \int e^u \cos u\, du = -e^u \cos u + e^u \sin u - \int e^u \sin u\, du$.

Therefore, $\int e^{3x} \sin 5x\, dx = \frac{1}{2}\left(-e^u \cos u + e^u \sin u\right) + C$.

29. Let $u = \sqrt{x}$ then $x = u^2$ and $dx = 2udu$.

$\int \sin\sqrt{x}\, dx = 2\int u \sin u\, du$ and the integral can be evaluated using integration by parts.

Let $w = u$ and $dv = \sin u du$ then $dw = du$ and $v = -\cos u$.

$2\int u \sin u\, du = -2u \cos u + 2\int \cos u\, du = -2u\cos u + 2\sin u + C.$

30. At first, let solve corresponding uniform the differential equation:

$xy' + y = 0,$

$\frac{dy}{dx} = -\frac{y}{x},$

$\frac{dy}{y} = -\frac{dx}{x},$

$y = \frac{C(x)}{x}.$

Then $y' = \frac{C'(x)x - C(x)}{x^2}$ and $x\frac{C'(x)x - C(x)}{x^2} + \frac{C(x)}{x} = xe^x.$

Thus $C(x) = xe^x - e^x + C$ and $y = \frac{xe^x - e^x + C}{x}$.

31.

a) Let $u = \sin^{n-1} ax$ and $dv = \sin ax dx$ then $du = a(n-1)\sin^{n-2} ax \cos ax$ and $v = -\frac{\cos ax}{a}$.

$$\int \sin^n ax\, dx = -\frac{\sin^{n-1} ax \cos ax}{a} + (n-1)\int \sin^{n-2} ax \cos^2 ax\, dx$$

$$= -\frac{\sin^{n-1} ax \cos ax}{a} + (n-1)\int \sin^{n-2} ax \cos ax(1-\sin^2 ax)\, dx$$

$$= -\frac{\sin^{n-1} ax \cos ax}{a} + (n-1)\int (\sin^{n-2} ax - \sin^n ax)\, dx.$$

Therefore, $\int \sin^n ax\, dx = -\frac{\sin^{n-1} ax \cos ax}{an} + \frac{n-1}{n}\int \sin^{n-2} ax\, dx$.

b) $\int \sin^6 3x\, dx = -\frac{\sin^5 3x \cos 3x}{3\cdot 6} + \frac{5}{6}\int \sin^4 3x\, dx = -\frac{\sin^5 3x \cos 3x}{18} - \frac{5}{6}\cdot\frac{\sin^3 3x \cos 3x}{3\cdot 4} + \frac{5}{6}\cdot\frac{3}{4}\int \sin^2 3x\, dx = -\frac{\sin^5 3x \cos 3x}{18} - \frac{5\sin^3 3x \cos 3x}{72} + \frac{5}{8}\int \frac{1-\cos 6x}{2}\, dx = -\frac{\sin^5 3x \cos 3x}{18}$

$-\dfrac{5\sin^3 3x\cos 3x}{72}+\dfrac{5}{8}\left(\dfrac{1}{2}x-\dfrac{\cos 6x}{12}\right)+C=-\dfrac{\sin^5 3x\cos 3x}{18}-\dfrac{5\sin^3 3x\cos 3x}{72}+\dfrac{5}{16}x-\dfrac{5\cos 6x}{96}+C.$

$\int\sin^7 4x\,dx=-\dfrac{\sin^6 4x\cos 4x}{4\cdot 7}+\dfrac{6}{7}\int\sin^5 4x\,dx=-\dfrac{\sin^6 4x\cos 4x}{28}-\dfrac{6}{7}\cdot\dfrac{\sin^4 4x\cos 4x}{4\cdot 5}+\dfrac{6}{7}\cdot\dfrac{4}{5}\int\sin^3 4x\,dx=-\dfrac{\sin^6 4x\cos 4x}{28}-\dfrac{6\sin^4 4x\cos 4x}{140}-\dfrac{24}{35}\cdot\dfrac{\sin^2 4x\cos 4x}{4\cdot 3}+\dfrac{2}{3}\cdot\dfrac{24}{35}\int\sin 4x\,dx=-\dfrac{\sin^6 4x\cos 4x}{28}-\dfrac{6\sin^4 4x\cos 4x}{140}-\dfrac{12\sin^2 4x\cos 4x}{35}+\dfrac{4\cos 4x}{35}+C.$

32.

a) Let $u=(\ln ax)^m$ and $dv=x^n\,dx$ then $du=\dfrac{m(\ln ax)^{m-1}}{x}$ and $v=\dfrac{x^{n+1}}{n+1}$.

Therefore, $\int x^n(\ln ax)^m\,dx=(\ln ax)^m\dfrac{x^{n+1}}{n+1}-\dfrac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$\int x^4(\ln 3x)^2\,dx=\dfrac{x^5(\ln 3x)^2}{5}-\dfrac{2}{6}\int x^4\ln 3x\,dx=\dfrac{x^5(\ln 3x)^2}{5}-\dfrac{2}{6}\left(\dfrac{x^5\ln 3x}{5}-\int\dfrac{x^4}{5}dx\right)=\dfrac{x^5(\ln 3x)^2}{5}-\dfrac{2x^5\ln 3x}{30}-\dfrac{2x^5}{150}+C.$

33. Let $u=\dfrac{1}{\cos^{n-2}ax}$ and $dv=\dfrac{1}{\cos^2 ax}dx$ then $du=-\dfrac{a(-n+2)\sin ax}{\cos^{n-1}ax}$ and $v=\dfrac{\tan ax}{a}$.

$\int\sec^n ax\,dx=-\dfrac{\tan ax}{a\cos^{n-2}ax}+(-n+2)\int\dfrac{\sin^2 ax}{\cos^n ax}dx=-\dfrac{\sin ax}{a\cos^{n-1}ax}+(-n+2)\int\sec^n ax\,dx-(-n+2)\int\sec^{n-2}ax\,dx+C.$

Therefore, $\int\sec^n ax\,dx=-\dfrac{\sin ax}{a(n-1)\cos^{n-1}ax}-\dfrac{2-n}{n-1}\int\sec^{n-2}ax\,dx+C.$

$\int\sec^3 2x\,dx=-\dfrac{\sin 2x}{4\cos^2 2x}+\dfrac{1}{2}\int\sec 2x\,dx=-\dfrac{\sin 2x}{4\cos^2 2x}+\dfrac{1}{4}\ln|\sec 2x+\tan 2x|+C.$

$\int\sec^4 2x\,dx=-\dfrac{\sin 2x}{6\cos^3 2x}+\dfrac{2}{3}\int\sec^2 2x\,dx=-\dfrac{\sin 2x}{6\cos^3 2x}+\dfrac{1}{3}\tan 2x+C.$

34.

a) Let $u=x^n$ and $dv=\sin ax\,dx$ then $du=nx^{n-1}$ and $v=\dfrac{-\cos ax}{a}$.

$\int x^n\sin ax\,dx=\dfrac{-x^n\cos ax}{a}+\dfrac{n}{a}\int x^{n-1}\cos ax\,dx=\dfrac{-x^n\cos ax}{a}+\dfrac{nx^{n-1}\sin ax}{a^2}-\dfrac{n(n-1)}{a^2}\int x^{n-2}\sin ax\,dx.$

b) Let $u=x^n$ and $dv=\cos ax\,dx$ then $du=nx^{n-1}$ and $v=\dfrac{\sin ax}{a}$.

$\int x^n\cos ax\,dx=\dfrac{x^n\sin ax}{a}-\dfrac{n}{a}\int x^{n-1}\sin ax\,dx=\dfrac{x^n\sin ax}{a}+\dfrac{nx^{n-1}\cos ax}{a^2}-\dfrac{n(n-1)}{a^2}\int x^{n-2}\cos ax\,dx.$

c) Let $u = \dfrac{1}{\sin^{n-2} ax}$ and $dv = \dfrac{1}{\sin^2 ax}\,dx$ then $du = \dfrac{a(-n+2)\cos ax}{\sin^{n-1} ax}$ and $v = \dfrac{-\cot ax}{a}$.

$\int \csc^n ax\ dx = \dfrac{-\cot ax}{a\sin^{n-2} ax} + (-n+2)\int \dfrac{\cos^2 ax}{\sin^n ax}dx = -\dfrac{\cos ax}{a\sin^{n-1} ax} + (-n+2)\int \csc^n ax\,dx -$
$(-n+2)\int \csc^{n-2} ax\,dx + C.$

Therefore, $\int \csc^n ax\ dx = -\dfrac{\cos ax}{a(n-1)\sin^{n-1} ax} + \dfrac{n-2}{n-1}\int \csc^{n-2} ax\,dx + C.$

$+\ (-n+2)\int \sec^n ax\,dx - (-n+2)\int \sec^{n-2} ax\,dx + C.$

d) Let $u = x^n$ and $dv = b^{ax}\ dx$ then $du = nx^{n-1}$ and $v = \dfrac{b^{ax}}{a\ln b}$.

$\int x^n b^{ax}\ dx = \dfrac{x^n b^{ax}}{a\ln b} - \dfrac{n}{a\ln b}\int x^{n-1} b^{ax}\ dx.$

e) Let $u = \left(\sqrt{x^2-a^2}\right)^n$ and $dv = dx$ then $du = nx\left(\sqrt{x^2-a^2}\right)^{n-2}$ and $v = x$.

$\int \left(\sqrt{x^2-a^2}\right)^n dx$

$= x\left(\sqrt{x^2-a^2}\right)^n - \int nx^2\left(\sqrt{x^2-a^2}\right)^{n-2} dx$

$= x\left(\sqrt{x^2-a^2}\right)^n - n\int (x^2-a^2)\left(\sqrt{x^2-a^2}\right)^{n-2} dx - na^2\int \left(\sqrt{x^2-a^2}\right)^{n-2} dx$

$= x\left(\sqrt{x^2-a^2}\right)^n - n\int \left(\sqrt{x^2-a^2}\right)^n dx - na^2\int \left(\sqrt{x^2-a^2}\right)^{n-2} dx.$

Therefore, $\int \left(\sqrt{x^2-a^2}\right)^n dx = \dfrac{1}{n+1}x\left(\sqrt{x^2-a^2}\right)^n - \dfrac{na^2}{n+1}\int \left(\sqrt{x^2-a^2}\right)^{n-2} dx.$

f) Let $u = \dfrac{1}{\left(\sqrt{x^2-a^2}\right)^n}$ and $dv = dx$ then $du = \dfrac{-nx}{\left(\sqrt{x^2-a^2}\right)^n}$ and $v = x$.

$\int \dfrac{dx}{\left(\sqrt{x^2-a^2}\right)^n} = \dfrac{x}{\left(\sqrt{x^2-a^2}\right)^n} + n\int \dfrac{x^2dx}{\left(\sqrt{x^2-a^2}\right)^n} = \dfrac{x}{\left(\sqrt{x^2-a^2}\right)^n} + n\int \dfrac{dx}{\left(\sqrt{x^2-a^2}\right)^{n-1}} +$
$na^2\int \dfrac{dx}{\left(\sqrt{x^2-a^2}\right)^n}.$

Therefore, $\int \dfrac{dx}{\left(\sqrt{x^2-a^2}\right)^n} = \dfrac{x}{(1-na^2)\left(\sqrt{x^2-a^2}\right)^n} + \dfrac{n}{1-na^2}\int \dfrac{dx}{\left(\sqrt{x^2-a^2}\right)^{n-1}}.$

4. Numerical Integration

1.

i-

a) a = 0, b = 2, h = 0.5, x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2.

$$I \approx I_4 = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

= 0.25[1 + 2(0.87758) + 2(0.54030) + 2(0.07074) + (–0.41615)]

= 0.89027.

b) $I = \int_0^2 \cos x \, dx = \sin x\big|_0^2$ = sin2 = 0.90930.

c) $f(x)$ = cos x, $f'(x)$ = –sin$x$, $f''(x)$ = –cos$x$. The maximum value M of $|f''|$ on [0, 2] is 1.

Using n = 4, h = 0.5 and $E_T \leq \left|\frac{(2-0)(0.5^2)}{12}(1)\right|$ = 0.04167.

d) Using n = 4 the relative error is $\frac{0.90930 - 0.89027}{0.90930}$ = 0.02093 which is less than E_T = 0.4167.

e) Using n = 4 the relative error is 0.02093 = 2.093%

ii- $f(x) = \frac{1}{x^2}$.

a) a = 1, b = 3, h = 0.5, x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3.

$$I \approx I_4 = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

= 0.25[1 + 2(0.44444) + 2(0.25) + 2(0.16) + (0.11111)]

= 0.70450.

b) $I = \int_1^3 \frac{1}{x^2}\, dx = \left.\frac{-1}{x}\right|_1^3 = \frac{-1}{3} + 1 = 0.66667.$

c) $f(x) = \frac{1}{x^2}$, $f'(x) = \frac{-2}{x^3}$, $f''(x) = \frac{6}{x^4}$. The maximum value M of $|f''|$ on [1, 3] is 6.

Using $n = 4$, $h = 0.5$ and $E_T \le \left|\frac{(3-1)(0.5^2)}{12}(6)\right| = 0.25.$

d) Using $n = 4$ the relative error is $\left|\frac{0.66667 - 0.70450}{0.66667}\right| = 0.05750$ which is less than $E_T = 0.25$.

e) Using $n = 4$ the relative error is 0.05750 = 5.750%

iii- $f(x) = \sin 2x$.

a) $a = 0$, $b = \pi$, $h = \frac{\pi}{4}$, $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$.

$$I \approx I_4 = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$= \frac{\pi}{8}\, [0 + 2(1) + 2(0) + 2(-1) + 0]$$

$$= 0.$$

b) $I = \int_0^\pi \sin 2x\, dx = \left.\frac{-\cos 2x}{2}\right|_0^\pi = \frac{-1}{2} + \frac{1}{2} = 0.$

c) $f(x) = \sin 2x$, $f'(x) = 2\cos 2x$, $f''(x) = -4\sin 2x$. The maximum value M of $|f''|$ on $[0, \pi]$ is 4.

Using $n = 4$, $h = \frac{\pi}{4}$ and $E_T \le \left|\frac{(\pi - 0)(\frac{\pi}{4})^2}{12}(4)\right| = 0.64594.$

d) The relative error is 0 which is less than $E_T = 0.64594$.

e) A percentage of the exact value is 0%.

iv- $f(x) = x^4 - 2x^2$.

a) $a = 0$, $b = 1$, $h = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$.

$$I \approx I_4 = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$= 0.125[0 + 2(-0.12109) + 2(-0.4375) + 2(-0.80859) + (-1)]$

$= -0.46680$.

b) $I = \int_0^1 \left(x^4 - 2x^2\right) dx = \left(\frac{x^5}{5} - \frac{2x^3}{3}\right)\Big|_0^1 = -0.46667$.

c) $f(x) = x^4 - 2x^2$, $f'(x) = 4x^3 - 4x$, $f''(x) = 12x^2 - 4$. The maximum value M of $|f''|$ on [0, 1] is 8.

Using $n = 4$, $h = 0.25$ and $E_T \le \left|\frac{(1-0)(0.25^2)}{12}(8)\right| = 0.04167$.

d) Using $n = 4$ the relative error is $\left|\frac{-0.46667-(-0.46680)}{-0.46667}\right| = 0.00028$ which is less than $E_T = 0.04167$.

e) Using $n = 4$ the relative error is 0.00028 = 0.028%

v- $f(x) = 4x - 3$.

a) $a = 1$, $b = 3$, $h = 0.5$, $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$.

$$I \approx I_4 = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$= 0.25[1 + 2(3) + 2(5) + 2(7) + (9)]$

$= 10$.

b) $I = \int_1^3 (4x - 3)\, dx = \left(2x^2 - 3x\right)\Big|_1^3 = 10$.

c) $f(x) = 4x - 3$, $f'(x) = 4$, $f''(x) = 0$. The maximum value M of $|f''|$ on [1, 3] is 0.

Using $n = 4$, $h = 0.5$ and $E_T \le \left|\frac{(3-1)(0.5^2)}{12}(0)\right| = 0$.

d) Using $n = 4$ the relative error is $\left|\frac{10-10}{10}\right| = 0$ which is equal to $E_T = 0$.

e) Using $n = 4$ the relative error is 0%.

vi- $f(x) = x^3 - 4x$.

a) $a = 1$, $b = 2$, $h = 0.25$, $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$.

$$I \approx I_4 = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$= 0.125[(-3) + 2(-3.04688) + 2(-2.625) + 2(-1.64063) + 0]$

$= -2.20313$.

b) $I = \int_1^2 \left(x^3 - 4x\right) dx = \left(\frac{x^4}{4} - 2x^2\right)\Big|_1^2 = -2.25$.

c) $f(x) = x^3 - 4x$, $f'(x) = 3x^2 - 4$, $f''(x) = 6x$. The maximum value M of $|f''|$ on [1, 2] is 12.

Using $n = 4$, $h = 0.25$ and $E_T \le \left|\frac{(2-1)(0.25^2)}{12}(12)\right| = 0.0625$.

d) Using $n = 4$ the relative error is $\left|\frac{-2.25-(-2.20313)}{-2.25}\right| = 0.02083$ which is less than E_T = 0.0625.

e) Using $n = 4$ the relative error is 0.02083 = 2.083%

2.

i-

a) $a = 0$, $b = 2$, $h = 0.5$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$.

$$I \approx I_4 = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

= 0.5/3[1 + 4(0.87758) + 2(0.54030) + 4(0.7074) + (–0.41615)]

= 0.90962.

b) $I = \int_0^2 \cos x \, dx = \sin x\Big|_0^2$ = sin2 = 0.90930.

c) $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(iv)}(x) = \cos x$. The maximum value M of $\left|f^{(iv)}\right|$ on [0, 2] is 1.

Using $n = 4$, $h = 0.5$ and $E_T \le \left|\frac{(2-0)(0.5^4)}{180}(1)\right| = 0.00069$.

d) Using $n = 4$ the relative error is $\frac{0.90930 - 0.90962}{0.90930} = 0.00035$ which is less than E_T = 0.00069.

e) Using $n = 4$ the relative error is 0.00035 = 0.035%

ii-

a) $a = 1$, $b = 3$, $h = 0.5$, $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$.

$$I \approx I_4 = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

= 0.5/3[1 + 4(0.44444) + 2(0.25) + 4(0.16) + (0.11111)]

= 0.67148.

b) $I = \int_1^3 \frac{1}{x^2} dx = \frac{-1}{x}\Big|_1^3 = \frac{-1}{3} + 1 = 0.66667$.

c) $f(x) = \frac{1}{x^2}$, $f' = \frac{-2}{x^3}$, $f''(x) = \frac{6}{x^4}$, $f'''(x) = \frac{-24}{x^5}$, $f^{(iv)}(x) = \frac{120}{x^6}$. The maximum value M of $\left|f^{(iv)}\right|$ on [1, 3] is 120.

Using n = 4, h = 0.5 and $E_T \le \left|\frac{(3-1)(0.5^4)}{180}(120)\right|$ = 0.08333.

d) Using n = 4 the relative error is $\left|\frac{0.66667-0.67148}{0.66667}\right|$ = 0.00721 which is less than E_T = 0.08333.

e) Using n = 4 the relative error is 0.00721 = 0.721%

iii-

a) $a = 0$, $b = \pi$, $h = \frac{\pi}{4}$, $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$.

$$I \approx I_4 = \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)]$$

$$= \frac{\pi}{12}\,[0 + 4(1) + 2(0) + 4(-1) + 0]$$

$$= 0.$$

b) $I = \int_0^{\pi} \sin 2x\, dx = \left.\frac{-\cos 2x}{2}\right|_0^{\pi} = \frac{-1}{2}+\frac{1}{2} = 0.$

c) $f(x) = \sin 2x$, $f'(x) = 2\cos 2x$, $f''(x) = -4\sin 2x$, $f'''(x) = -8\cos 2x$, $f^{(iv)}(x) = 16\sin 2x$.

The maximum value M of $\left|f^{(iv)}\right|$ on $[0, \pi]$ is 16.

Using n = 4, $h = \frac{\pi}{4}$ and $E_T \le \left|\frac{(\pi-0)(\frac{\pi}{4})^4}{180}(16)\right|$ = 0.10625.

d) The relative error is 0 which is less than E_T = 0.10625.

e) A percentage of the exact value is 0%.

iv-

a) $a = 0$, $b = 1$, $h = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$.

$I \approx I_4 = \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)]$

= 0.25/3[0 + 4(–0.12109) + 2(–0.4375) + 4(–0.80859) + (–1)]

= –0.46614.

b) $I = \int_0^1 \left(x^4 - 2x^2\right) dx = \left(\frac{x^5}{5} - \frac{2x^3}{3}\right)\Big|_0^1$ = –0.46667.

c) $f(x) = x^4 - 2x^2$, $f'(x) = 4x^3 - 4x$, $f''(x) = 12x^2 - 4$, $f'''(x) = 24x$, $f^{(iv)}(x) = 24$. The maximum value M of $\left|f^{(iv)}\right|$ on [0, 1] is 24.

Using $n = 4$, $h = 0.25$ and $E_T \le \left|\frac{(1-0)(0.25^4)}{180}(24)\right|$ = 5.20833.

d) Using $n = 4$ the relative error is $\left|\frac{-0.46667-(-0.46614)}{-0.46667}\right|$ = 0.00114 which is less than E_T = 5.20833.

e) Using $n = 4$ the relative error is 0.00114 = 0.114%

v-

a) $a = 1$, $b = 3$, $h = 0.5$, $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$.

$I \approx I_4 = \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)]$

= 0.16667[1 + 4(3) + 2(5) + 4(7) + (9)]

= 10.0002.

b) $I = \int_1^3 \left(4x-3\right) dx = \left(2x^2 - 3x\right)\Big|_1^3$ = 10.

c) $f(x) = 4x - 3$, $f'(x) = 4$, $f''(x) = f'''(x) = f^{(iv)}(x) = 0$. The maximum value M of $\left|f^{(iv)}\right|$ on [1, 3] is 0.

Using n = 4, h = 0.5 and $E_T \le \left|\frac{(3-1)(0.5^4)}{180}(0)\right|$ = 0.

d) Using n = 4 the relative error is $\left|\frac{10-10.0002}{10}\right|$ = 0.00002 which is more than E_T = 0.

e) Using n = 4 the relative error is 0.00002 = 0.002%.

vi-

a) a = 1, b = 2, h = 0.25, x_0 = 1, x_1 = 1.25, x_2 = 1.5, x_3 = 1.75, x_4 = 2.

$I \approx I_4 = \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)]$

= 0.25/3[(–3) + 4(–3.04688) + 2(–2.625) + 4(–1.64063) + 0]

= –2.25000.

b) $I = \int_1^2 \left(x^3-4x\right) dx = \left.\left(\frac{x^4}{4}-2x^2\right)\right|_1^2$ = –2.25.

c) $f(x) = x^3-4x$, $f'(x) = 3x^2-4$, $f''(x) = 6x$, $f'''(x)$ = 6, $f^{(iv)}(x)$ = 0. The maximum value M of $\left|f^{(iv)}\right|$ on [1, 2] is 0.

Using n = 4, h = 0.25 and $E_T \le \left|\frac{(2-1)(0.25^4)}{180}(0)\right|$ = 0.

d) Using n = 4 the relative error is $\left|\frac{-2.25-(-2.25000)}{-2.25}\right|$ = 0 which is equal to E_T .

e) Using n = 4 the relative error is 0%.

3.

a)

i- $f(x)$ = cos x, $f'(x)$ = –sin$x$, $f''(x)$ = –cos$x$, $f'''(x)$ = sin$x$.

The roots of $f'''(x)$ on [0, π] are 0 and π.

$f''(0) = -1$ and $f''(\pi) = 1$

The maximum value M of $|f''(x)|$ on $[0, \pi]$ is 1.

The error bound $\left|\frac{(b-a)h^2}{12}M\right|$ is required to be less than 10^{-5}. That is,

$\left|\frac{(\pi-0)h^2}{12}1\right| < 10^{-5}$ which is equivalent to $h < 0.00618$.

Solving for n in $\frac{\pi-0}{n} = h < 0.00618$ yields $n > 508.3333$.

Therefore one must use 509 subintervals to obtain an answer to within 10^{-5} from the exact answer.

ii- $f(x) = \frac{1}{x^2}$, $f'(x) = \frac{-2}{x^3}$, $f''(x) = \frac{6}{x^4}$, $f'''(x) = \frac{-24}{x^5}$.

There is no any root of $f'''(x)$ on [1, 3].

$f''(1) = 6$ and $f''(3) = 0.07407$

The maximum value M of $|f''(x)|$ on [1, 3] is 6.

The error bound $\left|\frac{(b-a)h^2}{12}M\right|$ is required to be less than 10^{-5}. That is,

$\left|\frac{(3-1)h^2}{12}6\right| < 10^{-5}$ which is equivalent to $h < 0.00316$.

Solving for n in $\frac{3-1}{n} = h < 0.00316$ yields $n > 632.511$.

Therefore one must use 633 subintervals to obtain an answer to within 10^{-5} from the exact answer.

iii-

$f(x) = \sin 2x$, $f'(x) = 2\cos 2x$, $f''(x) = -4\sin 2x$, $f'''(x) = -8\cos 2x$.

The only root of $f'''(x)$ on [0, π] is π/4.

$f''(0) = 0$, $f''(\frac{\pi}{4}) = -4$ and $f''(\pi) = 0$.

The maximum value M of $|f''(x)|$ on [0, π] is 4.

The error bound $\left|\frac{(b-a)h^2}{12}M\right|$ is required to be less than 10^{-5}. That is,

$\left|\frac{(\pi-0)h^2}{12}4\right| < 10^{-5}$ which is equivalent to $h < 0.00309$.

Solving for n in $\frac{\pi-0}{n} = h < 0.00309$ yields $n > 1016.69$.

Therefore one must use 1017 subintervals to obtain an answer to within 10^{-5} from the exact answer.

b)

i- $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(iv)}(x) = \cos x$, $f^{(v)}(x) = -\sin x$

The roots of $f^{(v)}(x)$ on [0, π] are 0 and π.

$f^{(iv)}(0) = 1$ and $f^{(iv)}(\pi) = -1$

The maximum value M of $|f^{(iv)}(x)|$ on [0, π] is 1.

The error bound $\left|\frac{(b-a)h^4}{180}M\right|$ is required to be less than 10^{-5}. That is,

$\left|\frac{(\pi-0)h^4}{180}1\right| < 10^{-5}$ which is equivalent to $h < 0.15471$.

Solving for n in $\frac{\pi-0}{n} = h < 0.15471$ yields $n > 20.3063$.

Therefore one must use 21 subintervals to obtain an answer to within 10^{-5} from the exact answer.

ii- $f(x) = \frac{1}{x^2}$, $f'(x) = \frac{-2}{x^3}$, $f''(x) = \frac{6}{x^4}$, $f'''(x) = \frac{-24}{x^5}$, $f^{(iv)}(x) = \frac{120}{x^6}$, $f^{(v)}(x) = \frac{-720}{x^7}$

There is no any root of $f^{(v)}(x)$ on [1, 3].

$f^{(iv)}(1) = 120$ and $f^{(iv)}(3) = 0.164609$.

The maximum value M of $\left|f^{(iv)}(x)\right|$ on [1, 3] is 120.

The error bound $\left|\frac{(b-a)h^4}{180}M\right|$ is required to be less than 10^{-5}. That is,

$\left|\frac{(3-1)h^4}{180}120\right| < 10^{-5}$ which is equivalent to $h < 0.05233$.

Solving for n in $\frac{3-1}{n} = h < 0.05233$ yields $n > 38.21899$.

Therefore one must use 39 subintervals to obtain an answer to within 10^{-5} from the exact answer.

iii-

$f(x) = \sin 2x$, $f'(x) = 2\cos 2x$, $f''(x) = -4\sin 2x$, $f'''(x) = -8\cos 2x$, $f^{(iv)}(x) = 16\sin 2x$, $f^{(v)}(x) = 32\cos 2x$.

The only root of $f^{(v)}(x)$ on [0, π] is π/4.

$f^{(iv)}(0) = 0$, $f^{(iv)}(\frac{\pi}{4}) = 16$ and $f^{(iv)}(\pi) = 0$.

The maximum value M of $\left|f^{(iv)}(x)\right|$ on [0, π] is 16.

The error bound $\left|\frac{(b-a)h^4}{180}M\right|$ is required to be less than 10^{-5}. That is,

$\left|\dfrac{(\pi-0)h^4}{180}16\right| < 10^{-5}$ which is equivalent to $h < 0.07736$.

Solving for n in $\dfrac{\pi-0}{n} = h < 0.07736$ yields $n > 40.61000$.

Therefore one must use 41 subintervals to obtain an answer to within 10^{-5} from the exact answer.

4.

a) $y = \ln(1 + \cos x)$, $y' = \dfrac{-\sin x}{1+\cos x}$, $y'' = \dfrac{-1}{1+\cos x}$, $y''' = \dfrac{-\sin x}{(1+\cos x)^2}$, $y^{(iv)} = \dfrac{-\cos x-\sin^2 x-1}{(1+\cos x)^3}$.

The only root of y''' on [0, 1] is 0.

$y''(0) = -\dfrac{1}{2}$ and $y''(1) = \dfrac{-1}{1+\cos 1} = -0.64922$.

The maximum value M of $|y''|$ on [0, 1] is 0.64922.

Since $-\cos x-\sin^2 x-1 = 0 \Leftrightarrow \cos x = 1$.

The only root of $y^{(v)}$ on [0, 1] is 0.

$y^{(iv)}(0) = -\dfrac{1}{4}$ and $y^{(iv)}(1) = \dfrac{-\cos 1-\sin^2 1-1}{1+\cos 1} = -1.45969$.

The maximum value M of $|y^{(iv)}|$ on [0, 1] is 1.45969.

b) The error bound $\left|\dfrac{(b-a)h^2}{12}M\right|$ is required to be less than 10^{-6}. That is,

$\left|\dfrac{(1-0)h^2}{12}0.64922\right| < 10^{-6}$ which is equivalent to $h < 0.004299$.

Solving for n in $\dfrac{1-0}{n} = h < 0.004299$ yields $n > 232.612235$.

Therefore one must use 233 subintervals to obtain an answer to within 10^{-6} from the exact answer.

The error bound $\left|\frac{(b-a)h^4}{180}M\right|$ is required to be less than 10^{-6}. That is,

$\left|\frac{(1-0)h^4}{180}1.45969\right| < 10^{-6}$ which is equivalent to $h < 0.105379$.

Solving for n in $\frac{1-0}{n} = h < 0.105379$ yields $n > 9.489557$.

Therefore one must use 10 intervals to obtain an answer to within 10^{-6} from the exact answer.

c) $a = 0$, $b = 1$, $h = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$.

$$I \approx I_4 = \frac{h}{3}[y(x_0) + 4y(x_1) + 2y(x_2) + 4y(x_3) + y(x_4)]$$

$$= 0.0833333[0.693147 + 4(0.677480) + 2(0.629985) + 4(0.549097) + (0.431979)]$$

$$= 0.607617$$

5.

ii- $f(x) = \frac{1}{x}$, $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = \frac{-6}{x^4}$, $f^{(iv)}(x) = \frac{24}{x^5}$, $f^{(v)}(x) = \frac{-120}{x^6}$.

There is no any root of $f^{(v)}(x)$ on [1, 2].

$f^{(iv)}(1) = 24$ and $f^{(iv)}(2) = 0.75$.

The maximum value M of $\left|f^{(iv)}(x)\right|$ on [1, 2] is 24.

The error bound $\left|\frac{(b-a)h^4}{180}M\right|$ is required to be less than 10^{-4}. That is,

$\left|\frac{(2-1)h^4}{180}24\right| < 10^{-4}$ which is equivalent to $h < 0.1655$.

Solving for n in $\frac{2-1}{n} = h < 0.1655$ yields $n > 6.0423$.

Therefore one must use 7 subintervals to obtain an answer to within 10^{-4} from the exact answer.

$\int_1^2 \frac{1}{x}\, dx = \ln x\Big|_1^2 = \ln 2.$

On the other hand for $a = 1$, $b = 2$, $h = \frac{1}{7}$, $x_0 = \frac{8}{7}$, $x_1 = \frac{9}{7}$, $x_2 = \frac{10}{7}$, $x_3 = \frac{11}{7}$, $x_4 = \frac{12}{7}$, $x_5 = \frac{13}{7}$, $x_6 = 2$ and $f(x) = \frac{1}{x}$ we have:

$$I \approx I_7 = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$= 0.0476[(0.875) + 4(0.7778) + 2(0.7) + 4(0.6364) + 2(0.5833) + 4(0.5385) + 0.5]$$

$$= 0.5594.$$

Therefore, ln2 = 0.5594 to for decimal places.

6. $f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = 4x^2e^{-x^2} - 2e^{-x^2}$.

As $|f''(x)| \leq 2$ on [0, 1], the maximum value M of $|f''(x)|$ on [0, 1] is 2.

The error bound $\left|\frac{(b-a)h^2}{12}M\right|$ is required to be less than 10^{-2}. That is,

$\left|\frac{(1-0)h^2}{12}2\right| < 10^{-2}$ which is equivalent to $h < 0.24$.

Solving for n in $\frac{1-0}{n} = h < 0.24$ yields $n > 4$.

Therefore one must use 5 subintervals to obtain an answer to within 10^{-2} from the exact answer.

$f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = 4x^2e^{-x^2} - 2e^{-x^2}$, $f'''(x) = 12xe^{-x^2} - 8x^3e^{-x^2}$, $f^{(iv)}(x) = 12e^{-x^2} - 48x^2e^{-x^2} + 16x^4e^{-x^2}$.

As $\left|f^{(iv)}(x)\right| \leq 12$ on [0, 1], the maximum value M of $\left|f^{(iv)}(x)\right|$ on [0, 1] is 12.

The error bound $\left|\dfrac{(b-a)h^4}{180}M\right|$ is required to be less than 10^{-2}. That is,

$\left|\dfrac{(1-0)h^4}{180}12\right| < 10^{-2}$ which is equivalent to $h < 0.62$.

Solving for n in $\dfrac{1-0}{n} = h < 0.62$ yields $n > 1.60$.

Therefore one must use 2 subintervals to obtain an answer to within 10^{-2} from the exact answer.

Using the trapezoidal method, we have
$a = 0$, $b = 1$, $h = 0.2$, $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$.

$$I \approx I_5 = \frac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+f(x_5)]$$

$$= 0.1[1 + 2(0.96) + 2(0.85) + 2(0.70) + 2(0.53) + 0.37]$$

$$= 0.75.$$

Using the trapezoidal method, we have
$a = 0$, $b = 1$, $h = 0.5$, $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$.

$$I \approx I_2 = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]$$

$$= 0.17[1 + 4(0.78) + 0.37]$$

$$= 0.76.$$

Practice Questions

1. Trigonometric Expressions

1. $\int_0^{\pi} \sin^2 4x \, dx = \int_0^{\pi} \frac{1-\cos 8x}{2} \, dx = \frac{1}{2}\left(x - \frac{\sin 8x}{8} \right)_0^{\pi} = \frac{\pi}{2}$.

2. $\int \sin^3 2\theta \cos^2 2\theta \, d\theta = \int \sin^2 2\theta \cos^2 2\theta \sin 2\theta d\theta = \int (1-\cos^2 2\theta)\cos^2 2\theta \sin 2\theta d\theta =$

$-\frac{1}{2}\left(\frac{\cos^3 2\theta}{3} - \frac{\cos^5 2\theta}{5} \right) + C = \frac{\cos^5 2\theta}{10} - \frac{\cos^3 2\theta}{6} + C$

3. $\int \frac{\sec^2(\ln x) \, dx}{x}$ $(u = \ln x)$

$= \int \sec^2 u du = \tan u + C = \tan(\ln x) + C.$

4. $\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx = \int_0^{\pi/2} \sin^4 x \cos^4 x \, dx = \frac{1}{16}\int_0^{\pi/2} (2\cos x \sin x)^4 \, dx = \frac{1}{16}\int_0^{\pi/2} \sin^4 2x \, dx =$

$\frac{1}{16}\int_0^{\pi/2} \left(\frac{1-\cos 4x}{2} \right)^2 dx = \frac{1}{64}\int_0^{\pi/2} \left(1 - 2\cos 4x + \cos^2 4x\right) dx = \frac{1}{64}\int_0^{\pi/2} \left(1 - 2\cos 4x + \frac{1+\cos 8x}{2} \right) dx$

$= \frac{1}{64}\int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 4x + \frac{\cos 8x}{2} \right) dx = \frac{1}{64}\left(\frac{3}{2}x - \frac{2\sin 4x}{4} + \frac{\sin 8x}{16} \right)_0^{\pi/2} = \frac{3\pi}{256}$.

5. $\int \tan^2 p \, dp = \int \frac{\sin^2 p}{\cos^2 p} dp$

$= \int \frac{(1-\cos^2 p)}{\cos^2 p} dp$

$= \int \left(\frac{1}{\cos^2 p} - 1 \right) dp$

$= (\tan p - p) + C.$

6. $\int_0^{\pi/2} \tan^3 2x \, dx = \int_0^{\pi/2} \frac{\sin^3 2x}{\cos^3 2x} dx$

$= \int_0^{\pi/2} \frac{\sin^2 2x \sin 2x}{\cos^3 2x} dx$

$= \int_0^{\pi/2} \frac{(1-\cos^2 2x)\sin 2x}{\cos^3 2x} dx$

$$= -\frac{1}{2}\left(-\frac{1}{2\cos^2 2x} - \ln|\cos 2x|\right)\Bigg|_0^{\pi/2}$$

$$= -\frac{1}{2}\left(-\frac{1}{2} - \ln 1 + \frac{1}{2} + \ln 1\right)\Bigg|_0^{\pi} = 0.$$

7. $\int e^x \sec^2\left(e^x\right) dx = \int \sec^2 u \; du$ $\qquad (u = e^x)$

$$= \tan u + C = \tan e^x + C$$

8. Let $x = 2\sin t$, then $dx = 2\cos t dt$.

$$\int \frac{\sqrt{4-x^2}}{3\,x}\, dx = \int \frac{\sqrt{4-4\sin^2 t}}{6\,\sin t}\, 2\cos t dt$$

$$= \frac{2}{3}\int \frac{\cos t}{\sin t} \cos t dt$$

$$= \frac{2}{3}\int \frac{\cos^2 t}{\sin t}\, dt$$

$$= \frac{2}{3}\int \frac{1-\sin^2 t}{\sin t}\, dt$$

$$= \frac{2}{3}\int (\frac{1}{\sin t} - \sin t) dt$$

$$= \frac{1}{3}\int \frac{1}{\sin\frac{t}{2}\cos\frac{t}{2}}\, dt - \frac{2}{3}\int \sin t dt$$

$$= \frac{1}{3}\int \frac{1}{\tan\frac{t}{2}\cos^2\frac{t}{2}}\, dt - \frac{2}{3}\int \sin t dt$$

$$= \frac{2}{3}\ln\left|\tan\frac{t}{2}\right| - \frac{2}{3}\cos t + C$$

$$= \frac{2}{3}\ln\left|\tan\left(\frac{1}{2}\sin^{-1}\frac{x}{2}\right)\right| - \frac{2}{3}\cos\left(\sin^{-1}\frac{x}{2}\right) + C$$

9. Let $x = \frac{7}{3}\tan t$, then $dx = \frac{7}{3\cos^2 t}dt$.

$$\int \frac{dx}{2x\sqrt{49+9x^2}} = \int \frac{\frac{7}{3\cos^2 t}dt}{2\cdot\frac{7}{3}\tan t\sqrt{49+49\tan^2 t}}$$

$$= \int \frac{\frac{1}{\cos^2 t} dt}{14 \frac{\sin t}{\cos t} \frac{1}{\cos t}}$$

$$= \int \frac{dt}{14 \sin t}$$

$$= \frac{1}{14} \int \csc t dt$$

$$= -\frac{1}{14} \ln |\csc t + \cot t| + C$$

$$= -\frac{1}{14} \ln \left| \csc\left(\tan^{-1} \frac{3}{7} x \right) + \cot\left(\tan^{-1} \frac{3}{7} x \right) \right| + C.$$

10. Let $x = \tan t$, then $dx = \frac{1}{\cos^2 t} dt$.

$$\int \frac{dx}{\left(1+x^2\right)^{1/2}} = \int \frac{dt}{\cos^2 t \left(1+\tan^2 t\right)^{1/2}}$$

$$= \int \frac{dt}{\cos^2 t \left(\frac{1}{\cos^2 t} \right)^{1/2}}$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C = \ln|\sec(\tan^{-1} x) + \tan(\tan^{-1} x)| + C$$

$$= \ln|\sqrt{1+x^2} + x| + C.$$

11. Let $\theta = \sec^{-1}|5x|$ then $|x| = \frac{1}{5}\sec\theta$ and $dx = \frac{1}{5}\sec\theta \tan\theta d\theta$ if $x \geq \frac{1}{5}$ and $dx = -\frac{1}{5}\sec\theta \tan\theta d\theta$ if $y \leq -\frac{1}{5}$.

$$\int \frac{dx}{x\sqrt{25x^2 - 1}} = \begin{cases} \int \frac{\frac{1}{5}\sec\theta \tan\theta d\theta}{\frac{1}{5}\sec\theta \sqrt{\sec^2\theta - 1}}, x \geq \frac{1}{25} \\ \int \frac{-\frac{1}{5}\sec\theta \tan\theta d\theta}{-\frac{1}{5}\sec\theta \sqrt{\sec^2\theta - 1}}, x \leq -\frac{1}{25} \end{cases}$$

$$= \int \frac{\tan\theta d\theta}{\tan\theta} = \theta + C = \sec^{-1}|5x| + C.$$

12. Let $\theta = \sec^{-1}\left|\frac{w}{5}\right|$ then $|w| = 5\sec\theta$ and $dw = 5\sec\theta \tan\theta d\theta$ if $w > 5$.

$$\int \frac{dw}{w^2\sqrt{w^2-25}} = \int \frac{5\cdot\sec\theta\tan\theta d\theta}{25\cdot\sec^2\theta\sqrt{25\sec^2\theta-25}} = \frac{1}{25}\int\frac{\tan\theta d\theta}{\sec\theta\tan\theta} = \frac{1}{25}\int\cos\theta d\theta = \frac{1}{25}\sin\theta + C$$

$$= \frac{\sqrt{x^2-25}}{25x} + C.$$

2. Partial Fractions

1.

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{Ax-A+Bx}{x(x-1)} = \frac{(A+B)x-A}{x(x-1)}.$$

This implies that $1 = (A + B)x - A$.

$$\begin{cases} A+B=0 \\ -A=1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=1 \end{cases}.$$

Therefore, $\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$.

$$\int\frac{dx}{x(x-1)} = \int\left(\frac{1}{x-1}-\frac{1}{x}\right)dx = \ln|x-1| - \ln|x| + C = \ln\left|\frac{x-1}{x}\right| + C.$$

2. $\frac{8}{(x^2-x)^2} = \frac{8}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} =$

$$\frac{Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2}{x^2(x-1)^2}.$$

This implies that $8 = Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2$.

The last equation is true for all $x\in\mathbb{R}$, even for $x = 0$ or $x = 1$.

Substitution $x = 0$ in $8 = Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2$ yields $B = 8$.

Substitution $x = 1$ in $8 = Ax(x-1)^2+B(x-1)^2+Cx^2(x-1)+Dx^2$ yields $D = 8$.

Then $8 = Ax(x-1)^2+8(x-1)^2+Cx^2(x-1)+8x^2$.

Substitution $x = 2$ in $8 = Ax(x-1)^2+8(x-1)^2+Cx^2(x-1)+8x^2$ yields $A + 2C = -16$.

Substitution $x = -1$ in $8 = Ax(x-1)^2+8(x-1)^2+Cx^2(x-1)+8x^2$ yields $2A + C = 16$.

The solution of two equations in A and C is $A = 16$ and $C = -16$.

Therefore, $\frac{8}{(x^2-x)^2} = \frac{16}{x} + \frac{8}{x^2} + \frac{-16}{x-1} + \frac{8}{(x-1)^2}$.

$$\int\frac{8\,dx}{(x^2-x)^2} = \int\left(\frac{16}{x}+\frac{8}{x^2}+\frac{-16}{x-1}+\frac{8}{(x-1)^2}\right)dx = 16\ln|x| - \frac{8}{x} - 16\ln|x-1| - \frac{8}{x-1} + C =$$

$$16\ln\left|\frac{x}{x-1}\right| - \frac{8}{x} - \frac{8}{x-1} + C.$$

3. $\dfrac{4}{x(x-1)(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1} = \dfrac{A(x-1)(x^2+1)+Bx(x^2+1)+(Cx+D)(x-1)x}{x(x-1)(x^2+1)} =$

$\dfrac{(A+B+C)x^3+(-A+D-C)x^2+(A+B-D)x-A}{x(x-1)(x^2+1)}$.

This implies that 4 = $(A+B+C)x^3+(-A+D-C)x^2+(A+B-D)x-A$.

$$\begin{cases} A+B+C=0 \\ -A+D-C=0 \\ A+B-D=0 \\ -A=4 \end{cases} \Rightarrow \begin{cases} A=-4 \\ B=2 \\ C=2 \\ D=0 \end{cases}.$$

Therefore, $\dfrac{4}{x(x-1)(x^2+1)} = \dfrac{-4}{x} + \dfrac{2}{x-1} + \dfrac{2x}{x^2+1}$.

$\displaystyle\int \frac{4\,dx}{x(x-1)(x^2+1)} = \int\left(\frac{-4}{x}+\frac{2}{x-1}+\frac{2x}{x^2+1}\right)dx = -4\ln|x| + 2\ln|x-1| + \ln(x^2+1) + C =$

$\ln\dfrac{(x^2+1)(x-1)^2}{x^4} + C$.

4. $\dfrac{1}{x^4+x^2} = \dfrac{1}{x^2(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1} = \dfrac{Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2}{x^2(x^2+1)} =$

$\dfrac{(A+C)x^3+(B+D)x^2+Ax+B}{x^2(x^2+1)}$.

This implies that 5 = $(A+C)x^3+(B+D)x^2+Ax+B$.

$$\begin{cases} A+C=0 \\ B+D=0 \\ A=0 \\ B=1 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=1 \\ C=0 \\ D=-1 \end{cases}.$$

Therefore, $\dfrac{1}{x^4+x^2} = \dfrac{1}{x^2} + \dfrac{-1}{x^2+1}$.

$\displaystyle\int \frac{dx}{x^4+x^2} = \int\left(\frac{1}{x^2}+\frac{-1}{x^2+1}\right)dx = -\frac{1}{x}-\tan^{-1}x + C.$

5. $\dfrac{8x+9}{(x+1)^3(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{D}{x+2} =$

$\dfrac{A(x+1)^2(x+2)+B(x+1)(x+2)+C(x+2)+D(x+1)^3}{(x+1)^3(x+2)}$.

This implies that $8x + 9 = A(x+1)^2(x+2)+B(x+1)(x+2)+C(x+2)+D(x+1)^3$.

The last equation is true for all $x \in \mathbb{R}$, even for $x = -2$ or $x = -1$.

Substitution $x = -2$ in $8x + 9 = A(x+1)^2(x+2) + B(x+1)(x+2) + C(x+2) + D(x+1)^3$ yields $D = 7$.

Substitution $x = -1$ in $8x + 9 = A(x+1)^2(x+2) + B(x+1)(x+2) + C(x+2) + D(x+1)^3$ yields $C = 1$.

Then $8x + 9 = A(x+1)^2(x+2) + B(x+1)(x+2) + (x+2) + 7(x+1)^3$.

Substitution $x = 0$ in $8x + 9 = A(x+1)^2(x+2) + B(x+1)(x+2) + (x+2) + 7(x+1)^3$ yields $A + B = 0$.

Substitution $x = 1$ in $8x + 9 = A(x+1)^2(x+2) + B(x+1)(x+2) + (x+2) + 7(x+1)^3$ yields $2A + B = -7$.

The solution of two equations in A and B is $A = -7$ and $B = 7$.

Therefore, $\dfrac{8x+9}{(x+1)^3(x+2)} = \dfrac{-7}{x+1} + \dfrac{7}{(x+1)^2} + \dfrac{1}{(x+1)^3} + \dfrac{7}{x+2}$.

$$\int \frac{(8x+9)\,dx}{(x+1)^3(x+2)} = \int\left(\frac{-7}{x+1} + \frac{7}{(x+1)^2} + \frac{1}{(x+1)^3} + \frac{7}{x+2}\right)dx = -7\ln|x+1| - \frac{7}{x+1} - \frac{1}{2(x+1)^2} + 7\ln|x+2| + C = 7\ln\left|\frac{x+2}{x+1}\right| - \frac{7}{x+1} - \frac{1}{2(x+1)^2} + C.$$

6. $\dfrac{20x+2}{(x-2)(x+1)(x+5)} = \dfrac{A}{x-2} + \dfrac{B}{x+1} + \dfrac{C}{x+5} =$

$$\frac{A(x+1)(x+5) + B(x-2)(x+5) + C(x-2)(x+1)}{(x-2)(x+1)(x+5)} =$$

$$\frac{(A+B+C)x^2 + (6A+3B-C)x + 5A-10B-2C}{(x-2)(x+1)(x+5)}.$$

This implies that $20x + 2 = (A+B+C)x^2 + (6A+3B-C)x + 5A - 10B - 2C$.

$$\begin{cases} A+B+C=0 \\ 6A+3B-C=20 \\ 5A-10B-2C=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=\dfrac{3}{2} \\ C=-\dfrac{7}{2} \end{cases}.$$

Therefore, $\dfrac{20x+2}{(x-2)(x+1)(x+5)} = \dfrac{2}{x-2} + \dfrac{\frac{3}{2}}{x+1} + \dfrac{-\frac{7}{2}}{x+5}$.

$$\int \frac{(20x+2)\,dx}{(x-2)(x+1)(x+5)} = \int\left(\frac{2}{x-2}+\frac{\frac{3}{2}}{x+1}+\frac{-\frac{7}{2}}{x+5}\right)dx = 2\ln|x-2|+\frac{3}{2}\ln|x+1|-\frac{7}{2}\ln|x+5|+C$$

$$= \ln\left(\frac{(x-2)^2\left(\sqrt{x+1}\right)^3}{\left(\sqrt{x+5}\right)^7}\right)+C.$$

3. Integration by Parts

1. Let $u = x$ and $dv = \cos 3x dx$ then $du = dx$ and $v = \dfrac{\sin 3x}{3}$.

$$\int 4x\cos 3x\ dx = \frac{4}{3}x\sin 3x - \frac{4}{3}\int \sin 3x\,dx = \frac{4}{3}x\sin 3x + \frac{4}{9}\cos 3x + C.$$

2. Let $u = \ln x$ and $dv = xdx$ then $du = \dfrac{dx}{x}$ and $v = \dfrac{x^2}{2}$.

$$\int_1^e x\ln x\ dx = \frac{x^2}{2}\ln x\Big|_1^e - \int_1^e \frac{x}{2}\,dx = \frac{x^2}{2}\ln x\Big|_1^e - \frac{x^2}{4}\Big|_1^e = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}.$$

3. Let $u = t^2$ and $dv = e^{-t}\ dt$ then $du = 2tdt$ and $v = -e^{-t}$.

$$\int_0^1 t^2 e^{-t}\ dt = -t^2e^{-t}\Big|_0^1 + 2\int_0^1 te^{-t}\ dt = -t^2e^{-t}\Big|_0^1 - 2te^{-t}\Big|_0^1 + 2\int_0^1 e^{-t}\ dt = -t^2e^{-t}\Big|_0^1 - 2te^{-t}\Big|_0^1 - 2e^{-t}\Big|_0^1$$

$= -e^{-1} - 2e^{-1} - 2e^{-1} + 2 = -5e^{-1} + 2.$

4. Let $u = e^{2x}$ and $dv = \sin 2x dx$ then $du = 2e^{2x}dx$ and $v = -\dfrac{\cos 2x}{2}$.

$$\int_0^\pi e^{2x}\sin 2x\ dx = -\frac{e^{2x}\cos 2x}{2}\Big|_0^\pi + \int_0^\pi e^{2x}\cos 2x\ dx = -\frac{e^{2x}\cos 2x}{2}\Big|_0^\pi + \frac{e^{2x}\sin 2x}{2}\Big|_0^\pi -$$

$$\int_0^\pi e^{2x}\sin 2x\ dx.$$

Therefore, $\displaystyle\int_0^\pi e^{2x}\sin 2x\ dx = -\frac{e^{2x}\cos 2x}{4}\Big|_0^\pi + \frac{e^{2x}\sin 2x}{4}\Big|_0^\pi = -\frac{e^{2\pi}}{4}+\frac{1}{4}.$

5. Let $u = \sin^{-1}x$ and $dv = dx$ then $du = \dfrac{dx}{\sqrt{1-x^2}}$ and $v = x$.

$$\int \sin^{-1}x\ dx = x\sin^{-1}x - \int \frac{xdx}{\sqrt{1-x^2}}\ dx = x\sin^{-1}x + \sqrt{1-x^2} + C.$$

6. Let $u = \ln x$ then $x = e^u$ and $dx = e^u du$.

$\int \cos(\ln x)\ dx = \int \cos u\, e^u\ du$ and the integral can be evaluated using integration by parts.

Let $w = e^u$ and $dv = \cos u du$ then $dw = e^u du$ and $v = \sin u$.

$$\int \cos u\, e^u\ du = e^u\sin u - \int e^u\sin u\ du = e^u\sin u + e^u\cos u - \int e^u\cos u\ du.$$

Therefore, $\displaystyle\int \cos(\ln x)\ dx = \frac{1}{2}\big(x\sin(\ln x)+x\cos(\ln x)\big)+C.$

7. Let $u = \ln x$ and $dv = dx$ then $du = \dfrac{dx}{x}$ and $v = x$.

$$\int_1^e \ln x\, dx = x\ln x\Big|_1^e - \int_1^e dx = x\ln x\Big|_1^e - x\Big|_1^e = e - e + 1 = 1.$$

8. Let $u = \tan^{-1} x$ and $dv = xdx$ then $du = \dfrac{dx}{1+x^2}$ and $v = \dfrac{x^2}{2}$.

$$\int x\tan^{-1} x\, dx = \frac{x^2\tan^{-1} x}{2} - \int \frac{x^2}{2(1+x^2)}dx = \frac{x^2\tan^{-1} x}{2} - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx = \frac{x^2\tan^{-1} x}{2} - \frac{x}{2}$$

$$+ \frac{\tan^{-1} x}{2} + C.$$

9. Let $u = \sec^{-1} x$ and $dv = xdx$ then $du = \dfrac{\sec^{-2} x}{\sqrt{1-x^2}}dx$ and $v = \dfrac{x^2}{2}$.

$$\int x\sec^{-1} x\, dx = \frac{x^2}{2}\sec^{-1} x - \int \frac{x^2\sec^{-2} x}{2\sqrt{1-x^2}}dx = \frac{x^2}{2}\sec^{-1} x + \frac{1}{2}\int\left(\sqrt{1-x^2}\sec^{-2} x - \frac{\sec^{-2} x}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^2}{2}\sec^{-1} x - \frac{1}{2}\sec^{-1} x + \frac{1}{2}\int\sqrt{1-x^2}\sec^{-2} x\, dx.$$

Let $u = 1-x^2$ and $dv = \dfrac{\sec^{-2} x}{\sqrt{1-x^2}}dx$ then $du = -2x$ and $v = -\sec^{-1} x$.

$$\frac{1}{2}\int\sqrt{1-x^2}\sec^{-2} x\, dx = -\frac{(1-x^2)}{2}\sec^{-1} x - \int x\sec^{-1} x\, dx + C.$$

Therefore, $\displaystyle\int x\sec^{-1} x\, dx = \frac{x^2}{4}\sec^{-1} x - \frac{1}{4}\sec^{-1} x - \frac{(1-x^2)}{4}\sec^{-1} x + C.$

10. Let $u = \cos^{n-1} ax$ and $dv = \cos ax dx$ then $du = -a(n-1)\cos^{n-2} ax\sin ax$ and $v = \dfrac{\sin ax}{a}$.

$$\int \cos^n ax\, dx = \frac{\sin ax\cos^{n-1} ax}{a} + (n-1)\int\sin^2 ax\cos^{n-2} ax\, dx$$

$$= \frac{\sin ax\cos^{n-1} ax}{a} + (n-1)\int\cos^{n-2} ax(1-\cos^2 ax)\, dx$$

$$= \frac{\sin ax\cos^{n-1} ax}{a} + (n-1)\int(\cos^{n-2} ax - \cos^n ax)\, dx.$$

Therefore, $\displaystyle\int \cos^n ax\, dx = \frac{\sin ax\cos^{n-1} ax}{an} + \frac{n-1}{n}\int\cos^{n-2} ax\, dx$.

11. At first, let solve corresponding uniform the differential equation:

$$y' + \frac{2}{x}y = 0,$$

$$\frac{dy}{dx} = -\frac{2y}{x},$$

$$\frac{dy}{2y} = -\frac{dx}{x},$$

$$y = \frac{C(x)}{x^2}.$$

Then $y' = \dfrac{C'(x)x^2 - 2xC(x)}{x^4}$ and $\dfrac{C'(x)x^2 - 2xC(x)}{x^4} + \dfrac{2}{x}\cdot\dfrac{C(x)}{x^2} = \cos x$.

Thus $C(x) = x^2\sin x + 2x\cos x - 2\sin x$ + C and $y = \dfrac{x^2\sin x + 2x\cos x - 2\sin x + C}{x^2}$.

12. Let $u = \cos^{m-1} ax$ and $dv = \cos ax \sin^n ax\, dx$ then $du = -a(m-1)\cos^{m-2} ax \sin ax$ and $v = \dfrac{\sin^{n+1} ax}{a(n+1)}$.

$\int \sin^n ax \cos^m ax\ dx = \cos^{m-1} ax \cdot \dfrac{\sin^{n+1} ax}{a(n+1)} + \dfrac{(m-1)}{(n+1)}\int \cos^{m-2} ax \sin^{n+2} ax dx = \cos^{m-1} ax \cdot \dfrac{\sin^{n+1} ax}{a(n+1)}$

$+ \dfrac{(m-1)}{(n+1)}\int \cos^{m-2} ax \sin^n ax(1-\cos^2 ax)dx = \cos^{m-1} ax \cdot \dfrac{\sin^{n+1} ax}{a(n+1)} +$

$\dfrac{(m-1)}{(n+1)}\int \cos^{m-2} ax \sin^n ax dx - \dfrac{(m-1)}{(n+1)}\int \cos^m ax \sin^n ax dx$.

Therefore, $\int \sin^n ax \cos^m ax\ dx = \cos^{m-1} ax \cdot \dfrac{\sin^{n+1} ax}{a(n+m)} + \dfrac{(m-1)}{(n+m)}\int \cos^{m-2} ax \sin^n ax dx$.

4. Numerical Integration

1.

a) $f(x) = 5x^4 - 3x^2$.

$a = 0$, $b = 2$, $h = \frac{1}{3}$, $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$, $x_4 = \frac{4}{3}$, $x_5 = \frac{5}{3}$, $x_6 = 2$.

$I \approx I_5 = \dfrac{h}{2}[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+f(x_6)]$

= 0.16667[0 + 2(–0.27160) + 2(–0.34568) + 2(2) + 2(10.46914) + 2(30.24692) + 68]

= 25.36677.

b) $I = \int_0^2 \left(5x^4 - 3x^2\right) dx = \left(x^5 - x^3\right)\Big|_0^2 = 24$.

c) $f(x) = 5x^4 - 3x^2$, $f'(x) = 20x^3 - 6x$, $f''(x) = 60x^2 - 6$. The maximum value M of $|f''|$ on [0, 2] is 234.

Using $n = 5$, $h = \frac{1}{3}$ and $E_T \le \left|\dfrac{(2-0)(1/3)^2}{12}(234)\right| = 4.33333$.

d) Using $n = 5$ the relative error is $\left|\dfrac{24 - 25.36676}{24}\right| = 0.05695$ which is less than $E_T = 4.33333$.

e) Using $n = 5$ the relative error is 0.05695 = 5.695%

2.

a) $f(x) = \dfrac{1}{x}$.

$a = 1$, $b = 2$, $h = 0.25$, $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$.

$I \approx I_4 = \dfrac{h}{2}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)]$

= 0.125[1 + 4(0.8) + 2(0.66667) + 4(0.57142) + 0.50000]

= 1.03988.

b) $I = \int_1^2 \dfrac{1}{x}\, dx = \ln x\Big|_1^2 = \ln 2 = 0.69315$.

c) $f(x) = \dfrac{1}{x}$, $f'(x) = -\dfrac{1}{x^2}$, $f''(x) = \dfrac{2}{x^3}$. The maximum value M of $|f''|$ on [1, 2] is 2.

Using n = 4, h = 0.25 and $E_S \le \left|\frac{(2-1)(0.25)^4}{180}(2)\right|$ = 0.00009.

d) Using n = 4 the relative error is $\left|\frac{0.69315-1.03988}{0.69315}\right|$ = 0.50022 which is more than E_S = 0.00009.

e) Using n = 5 the relative error is 0.50022 = 50.022%

3.

a) $f(x) = \cos\frac{x}{4}$, $f'(x) = -\frac{1}{4}\sin\frac{x}{4}$, $f''(x) = -\frac{1}{16}\cos\frac{x}{4}$, $f'''(x) = \frac{1}{64}\sin\frac{x}{4}$.

The roots of $f'''(x)$ on [0, π] are 0 and π.

$f''(0) = -1$ and $f''(\pi) = \frac{\sqrt{2}}{128}$.

The maximum value M of $|f''(x)|$ on [0, π] is 1.

The error bound $\left|\frac{(b-a)h^2}{12}M\right|$ is required to be less than 10^{-5}. That is,

$\left|\frac{(\pi-0)h^2}{12}1\right| < 10^{-5}$ which is equivalent to h < 0.06180.

Solving for n in $\frac{\pi-0}{n} = h$ < 0.06180 yields n > 50.83333.

Therefore one must use 51 subintervals to obtain an answer to within 10^{-5} from the exact answer.

b) $f(x) = \cos\frac{x}{4}$, $f'(x) = -\frac{1}{4}\sin\frac{x}{4}$, $f''(x) = -\frac{1}{16}\cos\frac{x}{4}$, $f'''(x) = \frac{1}{64}\sin\frac{x}{4}$, $f^{(iv)}(x) = \frac{1}{256}\cos\frac{x}{4}$, $f^{(v)}(x) = -\frac{1}{1024}\sin\frac{x}{4}$

The roots of $f^{(v)}(x)$ on [0, π] are 0 and π.

$f^{(iv)}(0) = 1$ and $f^{(iv)}(\pi) = -\frac{\sqrt{2}}{2048}$

The maximum value M of $|f^{(iv)}(x)|$ on [0, π] is 1.

The error bound $\left|\frac{(b-a)h^4}{180}M\right|$ is required to be less than 10^{-5}. That is,

$\left|\frac{(\pi-0)h^4}{180}1\right| < 10^{-5}$ which is equivalent to $h < 0.15471$.

Solving for n in $\frac{\pi-0}{n} = h < 0.15471$ yields $n > 20.3057$.

Therefore one must use 21 subintervals to obtain an answer to within 10^{-5} from the exact answer.

Test

Part I. Circle the correct answer.

1. $\int \cos^2 x \; \sin x \; dx \; =$

a) $-\frac{1}{3}\cos^3 x \; \sin x + C$

b) $-\frac{1}{3}\cos^3 x + C$

c) $-\frac{1}{2}\sin^2 x + C$

d) $\frac{1}{2}\sin^2 x + C$

2. $\int \cos^3 x \; \sin^2 x \; dx \; =$

a) $\int \left(u^4 - u^2\right) du \qquad (u = \cos x)$

b) $\int \left(u^4 - u^2\right) du \qquad (u = \sin x)$

c) $\int \left(u^2 - u^4\right) du \qquad (u = \cos x)$

d) $\int \left(u^2 - u^4\right) du \qquad (u = \sin x)$

3. The substitution $\theta = \sin^{-1}\frac{x}{3}$ transforms $\int \frac{x^3}{\sqrt{9-x^2}}\, dx$ to:

a) $-\int 27\cos^3\theta \, d\theta$

b) $\int 27\sin^3\theta \, d\theta$

c) $\int 9\cos^3\theta \, d\theta$

d) $\int 9\sin^3\theta \, d\theta$

4. $\int \sec(2x)\, dx \; =$

a) $\frac{1}{2}\ln|\sec x + \tan x| + C$

b) $\frac{1}{2}\ln|\sec 2x + \tan 2x| + C$

c) $\frac{1}{2}\tan^2(2x) + C$

d) $\frac{1}{2}\sec^2(2x) + C$

5. $\int x^3 e^{2x} \; dx \; =$

a) $\frac{1}{8}\left(4x^3 + 6x^2 + 6x + 3\right)e^{2x} + C$

b) $\left(x^3 - 3x^2 + 6x - 6\right)e^{2x} + C$

c) $\frac{1}{8}\left(4x^3 - 6x^2 + 6x - 3\right)e^{2x} + C$

d) $\frac{1}{8}\left(4x^3 - 6x^2 - 6x + 3\right)e^{2x} + C$

6. If $\frac{6x-1}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1}$ then

a) $A = 5$ and $B = -4$

b) $A = 5$ and $B = 6$

c) $A = -5$ and $B = -6$

d) $A = -5$ and $B = 4$

7. $\dfrac{(x+1)^2}{x\left(x^2+1\right)} =$

a) $\dfrac{2}{x}+\dfrac{1}{x^2+1}$ b) $\dfrac{1}{x}+\dfrac{2}{x^2+1}$

c) $\dfrac{1}{x}+\dfrac{2x+1}{x^2+1}$ d) $\dfrac{1}{x}-\dfrac{2x+1}{x^2+1}$

8. $\displaystyle\int \frac{(x+1)^2}{x\left(x^2+1\right)}\,dx =$

a) $\ln|x| + 2\tan^{-1}x + C$

b) $\ln|x| + \ln\left(x^2+1\right) + \tan^{-1}x + C$

c) $\ln|x| - \ln\left(x^2+1\right) - \tan^{-1}x + C$

d) $2\ln|x| + \tan^{-1}x + C$

9. $\displaystyle\int_1^{\sqrt{3}} \frac{(x+1)^2}{x\left(x^2+1\right)}\,dx$

a) $\dfrac{\pi}{12}+\dfrac{1}{2}\ln 3 - \ln 2$

b) $\dfrac{\pi}{12}+\ln 3$

c) $\dfrac{\pi}{12}+\dfrac{1}{2}\ln 3 + \ln 2$

d) $\dfrac{\pi}{6}+\dfrac{1}{2}\ln 3$

10. $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx =$

a) 0.5π

b) π

c) 2π

d) 4π

11. The error bound for the approximation of $\displaystyle\int_0^2 \frac{x-1}{x+1}\,dx$ using the trapezoidal rule with $n = 10$ subdivisions is:

a) 0.2667

b) 0.02667

c) 0.002667

d) 0.002667

12. The error bound for the approximation of $\displaystyle\int_0^2 \frac{x-1}{x+1}\,dx$ using Simpson's rule with $n =$ 10 subdivisions is:

a) 0.000085

b) 0.0085

c) 0.00085

d) 0.085

Part II. Show your work.

1. Evaluate $\int_0^{\pi/4} \cos^2 x \ \sin^3 x \ dx$

2. Evaluate $\int_0^4 \frac{x^3 \, dx}{\sqrt{25-x^2}}$

3. Use trigonometric substitution to derive the rule $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

4. Expand $\frac{9x-16}{x(x-4)(x+1)}$ using partial fractions.

5. Evaluate $\int \frac{4x^3+8x-4}{x^2(x^2+4)} dx$.

6. Evaluate $\int \frac{4x^3+8x-4}{x(x^2+4)} dx$.

7. Find the area of the region bounded by the curve $y = \ln(2x)$ and the x-axis from $x = 1$ to $x = 2$.

8. Find the volume of the solid obtained by revolving, about the y-axis, the region bounded by the curve $y = \ln(2x)$ and the x-axis from $x = 1$ to $x = 2$.

9. Find the volume of the solid obtained by revolving, about the x-axis, the region bounded by the curve $y = \ln(2x)$ and the x-axis from $x = 1$ to $x = 2$.

10. Consider the integral $I = \int_1^2 \ln(2x) \, dx$.

a) Find an approximation to I using the trapezoidal rule with $n = 4$ subdivision.

b) Find a bound on the error in the estimate obtained in a).

c) Find an approximation to I using Simpson's rule with $n = 4$ subdivision.

d) Find a bound on the error in the estimate obtained in c).

e) Find the exact value of I and the exact errors in the estimates in a) and c). Compare the exact errors with the bounds obtained in b) and d)

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