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MATHEMATICS: differential equations: methodological workshop on problem solving

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Annotation: The methodological workshop is given curriculum for the discipline "Mathematics" or "High Mathematics" of the professional training of the bachelor of non-mathematical specialties. The methodological workshop contains summary of lecture material and typical examples along with their solutions on some specific topics from Content module "Differential equations". The methodological workshop offers also exercises for students for solving by themselves. It could be useful in preparation for practical classes as well as the independent work and also for quizzes and the final test or exam.

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Inrtoduction

In Mathematics, a differential equation is an equation that contains one or more functions with its derivatives. The derivatives of the function define the rate of change of a function at a point. It is mainly used in fields such as physics, engineering, biology and so on. The primary purpose of the differential equation is the study of solutions that satisfy the equations and the properties of the solutions.

One of the easiest ways to solve the differential equation is by using explicit formulas. In this workshop, let us discuss the definition, types, methods to solve the differential equation, order and degree of the differential equation, ordinary differential equations with real-word examples and a solved problem.

Differential Equations can describe how populations change, how heat moves, how springs vibrate, how radioactive material decays and much more. They are a very natural way to describe many things in the universe.

So we try to solve them by turning the Differential Equation into a simpler equation without the differential bits, so we can do calculations, make graphs, predict the future, and so on.

Lecture summary

An equation that involves a function y of x along with its derivatives is called a differential equation. $y' + 2y = \sin x$, $y'' + xy' + 2y = x^2 + 1$, and $y'^2 + (\tan x)\, y = 1$ are all examples of *differential equations*.

- To solve $y' = f(x)$ integrate both sides: $y = \int f(x)\, dx$
- To solve $\dfrac{dy}{dx} = f(x)g(y)$ express as $\dfrac{dy}{g(y)} = f(x)dx$ and integrate: $\displaystyle\int \frac{dy}{g(y)} = \int f(x)dx$.
- To solve $y' + p(x)y = f(x)$ find $v(x) \equiv e^{\int p(x)dx}$ and set $y = \dfrac{1}{v(x)} \int f(x)\, v(x)\, dx$.

The conditions $y = y_0$, $y' = y'_0$, $y'' = y''_0$, etc. when $x = a$ are called *initial conditions* and the problem of solving the differential equation in this case is referred to as solving an *initial-value problem*. To solve an initial-value problem, find the general solution of the differential equation and then substitute the initial conditions to obtain the particular values of the constants involved.

The general form of a *second order linear differential equation* is $y'' + p(x)y' + q(x)y = f(x)$.

It is called *homogenous* if $f(x) \equiv 0$.

$ay'' + by' + cy = 0$, $a, b, c \in \mathbb{R}$ is a *homogeneous second order linear differential equation with constant coefficients*. Its *characteristic equation* is $ar^2 + br + c = 0$.

- If the characteristic equation has two distinct roots r_1 and r_2 then the general solution is $c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
- If the characteristic equation has two equal roots $r = r_1 = r_2 = -\dfrac{b}{2a}$ then the general solution is $c_1 e^{rx} + c_2 x e^{rx}$.
- If the characteristic equation has two complex roots $\alpha + \beta i$ and $\alpha - \beta i$ then the general solution is $y = e^{\alpha x}\left(c_1 \cos \beta x + c_2 \sin \beta x\right)$.

The differential equation $y'' + p(x)y' + q(x)y = f(x)$ is called *non-homogenous* if $f \neq 0$.
If $c_1u(x) + c_2v(x)$ is the general solution to $y'' + p(x)y' + q(x)y = 0$ and y_P *is a particular solution* to the equation, then the general solution is $y = c_1u(x) + c_2v(x) + y_P$.

For the differential equation $ay'' + by' + cy = f(x)$, a, b, and c are constants, the particular solution can be found using the *method of undetermined coefficients* as follows:

$f(x)$	y_P
$p_n(x)$, polynomial of degree n	$x^s q_n(x)$ s is the number of times 0 is a root of the characteristic equation.
$p_n(x)e^{rx}$	$x^s q_n(x)e^{rx}$ s is the number of times α is a root of the characteristic equation.
$f(x)$ is $p_n(x)e^{\alpha x}\begin{cases}\sin \beta x \\ \cos \beta x\end{cases}$	$x^s[g_n(x)e^{\alpha x}\sin \beta x + h_n(x)e^{\alpha x}\cos \beta x]$, s is the number of times $\alpha + \beta i$ is a root of the characteristic equation.

$q_n(x)$, $g_n(x)$, and $h_n(x)$ are polynomials of degree n whose coefficients are to be determined.

Section: 1 First Order Differential Equations

Example 1

Given that $y = 1$ at $x = 0$, solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = \sec^3 x$, giving your answer in the form $y = f(x)$.

Solution

$$\int -\tan x\,\mathrm{d}x = \ln|\cos x|$$

$$v(x) = \cos x$$

$$y = \frac{1}{\cos x}\int \cos x \frac{1}{\cos^3 x}\mathrm{d}x$$

$$y = \sec x(\tan x + \mathrm{C})$$

$$y = 1 \text{ at } \mathrm{x} = 0 \Rightarrow y = \sec x \tan x + \sec x$$

Example 2

a) Find $\int \frac{\mathrm{d}x}{\sqrt{9x^2+4}}$.

b) Find the general solution of the differential equation $(9x^2+4)\frac{\mathrm{d}y}{\mathrm{d}x} + 9xy = 1$

Solution

$$\int \frac{\mathrm{d}x}{\sqrt{9x^2+4}} = \frac{1}{3}\sinh^{-1}\left(\frac{3x}{2}\right) + \mathrm{C}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{9x}{9x^2+4}y = \frac{1}{9x^2+4}$$

$$v(x) = \sqrt{9x^2+4}$$

$$y = \frac{1}{\sqrt{9x^2+4}}\int \sqrt{9x^2+4}\frac{\mathrm{d}x}{9x^2+4}$$

$$= \frac{1}{\sqrt{9x^2+4}}\left[\frac{1}{3}\sinh^{-1}\left(\frac{3x}{2}\right) + \mathrm{C}\right]$$

Example 3

a) Show that the transformation $y = vx$ transforms the equation:

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(2 + 9x^2\right)y = x^5, \quad \text{I}$$

into the equation : $\frac{d^2v}{dx^2} + 9v = x^2, \quad \text{II}$

b) Solve the differential equation II to find v as a function of x.

c) Hence state the general solution of the differential equation I.

Solution

$$\frac{dy}{dx} = v + x\frac{dv}{dx}, \quad \frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

$$\Rightarrow x^2\left(2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right) - 2x\left(v + x\frac{dv}{dx}\right) + \left(2 + 9x^2\right)(vx) = x^5$$

$$\Rightarrow 9x^3v + x^3\frac{d^2v}{dx^2} = x^5$$

$$\Rightarrow 9v + \frac{d^2v}{dx^2} = x^2$$

$$r^2 + 9 = 0 \Rightarrow r = \mp 3i$$

$$\Rightarrow v_h = A\cos 3x + B\sin 3x$$

$$v_p = Cx^2 + Dx + E$$

$$\Rightarrow C = \frac{1}{9}, \quad E = \frac{-2}{81} \text{ and } D = 0$$

$$\Rightarrow v = A\cos 3x + B\sin 3x + \frac{1}{9}x^2 - \frac{2}{81}$$

$$y = \frac{2}{81}x + \frac{1}{9}x^3 + x\left(A\cos 3x + B\sin 3x\right)$$

Example 4

a) Show that the substitution $y = vx$ transforms the differential equation $\frac{dy}{dx} = \frac{3x - 4y}{4x + 3y} \quad \text{(I)}$

into the differential equation $x\frac{dv}{dx} = -\frac{3v^2+8v-3}{3v+4}$ (II).

b) By solving differential equation (II), find a general solution of differential equation (I).

c) Given that $y = 7$ at $x = 1$, show that the particular solution of differential equation (I) can be written as $(3y - x)(y + 3x) = 200$.

Solution

$$y = vx \Rightarrow y' = xv' + v = \frac{3x-4vx}{4x+3vx} = \frac{3-4v}{4+3v}$$

$$\Rightarrow xv' = \frac{3-4v}{4+3v} - v = \frac{3-4v}{4+3v} - \frac{4v+3v^2}{4+3v} = -\frac{-3+8v+3v^2}{4+3v}$$

$$\int \frac{3v+4}{3v^2+8v-3}dv = \int -\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{2}\ln\left|3v^2+8v-3\right| = \ln\left|\frac{A}{x}\right|$$

$$\Rightarrow 3v^2+8v-3 = \frac{B}{x^2}$$

$$\Rightarrow 3y^2+8xy-3x^2 = B$$

$y = 7$ at $x = 1 \Rightarrow (21-1)(7+3) = B = 200$

$$\Rightarrow (3y-x)(y+3x) = 200$$

Example 5

Given that $x = \ln t$, $t > 0$ and that y is a function of x,

a) Find $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and t,

b) Show that $\frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$.

c) Show that the substitution $x = \ln t$ transforms the differential equation

$\frac{d^2y}{dx^2} - (1-6e^x)\frac{dy}{dx} + 10ye^{2x} = 5e^{2x}\sin 2e^x$ (I) into the differential equation

$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 5\sin 2t.$ (II)

d) Hence find the general solution of (I), giving your answer in the form $y = f(x)$.

Solution

$$\frac{dy}{dx} = t\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = t\frac{d}{dt}\left(\frac{dy}{dx}\right) = t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right) = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

$$t^2y'' + ty' - (1-6t)(ty') + (10y)(t^2) = 5t^2\sin(2t)$$

$$\Rightarrow y'' + 6y' + 10y = 5\sin(2t)$$

$$r^2 + 6r + 10 = 0 \Rightarrow r = -3 \mp i$$

$$\Rightarrow y = e^{-3t}[A\cos t + B\sin t]$$

$$y_p = M\sin 2t + N\cos 2t$$

$$y_p' = 2M\cos 2t - 2N\sin 2t$$

$$y_p'' = -4M\sin 2t - 4N\cos 2t$$

Substituting into equation we get $M = \frac{1}{6}$, $N = -\frac{1}{3}$

$$\Rightarrow y = e^{-3t}[A\cos t + B\sin t] + \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t \quad , \quad t = e^x$$

Example 6

a) Show that $y = \frac{1}{2}x^2e^x$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

b) Solve the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$, given that $y = 1$ and $\frac{dy}{dx} = 2$ at $x = 0$.

Solution

$$y = \frac{1}{2}x^2e^x$$

$$y' = x\mathrm{e}^{x} + \frac{1}{2}x^{2}\mathrm{e}^{x}$$

$$y'' = \mathrm{e}^{x} + 2x\mathrm{e}^{x} + \frac{1}{2}x^{2}\mathrm{e}^{x}$$

$$\mathrm{e}^{x}\left(1 + 2x + \frac{1}{2}x^{2}\right) - 2\left(x + \frac{1}{2}x^{2}\right)\mathrm{e}^{x} + \frac{1}{2}x^{2}\mathrm{e}^{x} = \mathrm{e}^{x}$$

$\Rightarrow \mathrm{e}^{x} = \mathrm{e}^{x} \Rightarrow \frac{1}{2}x^{2}\mathrm{e}^{x}$ **is a solution of the differential equation**

$$r^{2} - 2r + 1 = 0$$
$$(r-1)^{2} = 0$$
$$r = 1$$

$$y = \left(A + Bx + \frac{1}{2}x^{2}\right)\mathrm{e}^{x}$$

$$y = 1, x = 0 \Rightarrow A = 1$$

$$y' = 2, x = 0 \Rightarrow B = 1$$

Example 7

a) Find the general solution of the differential equation $\frac{\mathrm{d}^{2}y}{\mathrm{d}t^{2}} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 2\mathrm{e}^{-t}$.

b) Find the particular solution that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$ at $t = 0$.

<u>Solution</u>

$$r^{2} + 2r + 2 = 0$$

$$r = -1 \pm i$$

$$\Rightarrow y_{h} = \mathrm{e}^{-t}\left(A\cos t + B\sin t\right)$$

$$y_{p} = M\mathrm{e}^{-t}$$

$$y_{p}' = -M\mathrm{e}^{-t}$$

$y_p'' = \mathrm{M}e^{-t}$

$\Rightarrow \mathrm{M} = 2$

$$\Rightarrow y = e^{-t}\left(\mathrm{A}\cos t + \mathrm{B}\sin t\right) + 2e^{-t} = e^{-t}\left(\mathrm{A}\cos t + \mathrm{B}\sin t + 2\right)$$

$y = 1$, t = 0 $\Rightarrow 1 = \mathrm{A} + 2 \Rightarrow \mathrm{A} = -1$

$y' = 1,\ t = 0 \Rightarrow 1 = -\mathrm{A} + \mathrm{B} - 2 \Rightarrow \mathrm{B} = 2$

$$\Rightarrow y = e^{-t}\left(-\cos t + 2\sin t + 2\right)$$

Example 8

Use the substitution $y = vx$, where v is a function of x, to solve the differential equation

$xy\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$, $x, y > 0$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$$

$$\Rightarrow x^2v\left(x\frac{\mathrm{d}v}{\mathrm{d}x} + v\right) = x^2 + v^2x^2$$

$$\Rightarrow vx^3\frac{\mathrm{d}v}{\mathrm{d}x} = x^2$$

$$\Rightarrow v\mathrm{d}v = \frac{\mathrm{d}x}{x}$$

$$\Rightarrow \frac{v^2}{2} = \ln x + \mathrm{C}$$

$$\Rightarrow y^2 = 2x^2\ln x + \mathrm{C}x^2$$

$$\Rightarrow y = x\sqrt{2\ln x + \mathrm{C}}$$

Example 9

Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}+2y\cot 2x=\sin x,\quad 0<x<\frac{\pi}{2}$$

<u>Solution</u>

$$\rho=\mathrm{e}^{\int 2\cot 2x\mathrm{d}x}=\mathrm{e}^{\ln(\sin 2x)}=\sin 2x$$

$$\Rightarrow y\sin 2x=\int \sin 2x\sin x\ \mathrm{d}x+\mathrm{C}$$

$$y\sin 2x=\frac{2}{3}\sin^3 x+\mathrm{C}$$

$$\Rightarrow y=\frac{1}{3}\tan x\sin x+\mathrm{C}\csc 2x,\quad \mathrm{C}\in\mathbf{R}$$

Example 10

Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}-y\tan x=\sec^2 x$, $0<x<\frac{\pi}{2}$, giving your answer for y in terms of x.

<u>Solution</u>

$$\rho=\mathrm{e}^{-\int \tan x\mathrm{d}x}$$

$$=\mathrm{e}^{\ln(\cos x)}=\cos x$$

$$\Rightarrow y\cos x=\int \sec^2 x\cos x\mathrm{d}x+\mathrm{C}$$

$$=\ln|\sec x+\tan x|+\mathrm{C}$$

$$\Rightarrow y=\sec x\left(\ln|\sec x+\tan x|+\mathrm{C}\right),\ \mathrm{C}\in\mathbb{R}$$

Example 11

Use the substitution $y=vx$, where v is a function of x, to transform the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+xy+y^2$, $x>0$ into $x\frac{\mathrm{d}v}{\mathrm{d}x}=1+v^2$, $x>0$. Hence, given that $y=\sqrt{3}$ at $x=1$, find y in terms of x.

Solution

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\Rightarrow x^2v+x^3\frac{dv}{dx}=x^2+x^2v+v^2x^2$$

$$\Rightarrow x\frac{dv}{dx}=1+v^2$$

$$\Rightarrow \int\frac{dv}{1+v^2}=\int\frac{dx}{x}\Rightarrow \tan^{-1}v=\ln x+C$$

$$\Rightarrow \tan^{-1}\frac{y}{x}=\ln x+C$$

$$y=\sqrt{3}\text{ at }x=1\Rightarrow \tan^{-1}\sqrt{3}=C$$

$$\Rightarrow \tan^{-1}\frac{y}{x}=\ln x+\tan^{-1}\sqrt{3}$$

$$\Rightarrow \boxed{y=x\tan\left(\ln x+\frac{\pi}{3}\right)}$$

Example 12

Find the general solution of the differential equation $\sin x\frac{dy}{dx}-y\cos x=\sin 2x\sin x$, giving your answer in the form $y=f\left(x\right)$.

Solution

$$\frac{dy}{dx}-\left(\cot x\right)y=\sin 2x$$

$$\rho=e^{-\int\cot x\,dx}=e^{-\ln|\sin x|}=\frac{1}{\sin x}$$

$$\Rightarrow \frac{1}{\sin x}y=\int\left(\sin 2x\right)\left(\frac{1}{\sin x}\right)dx+C$$

$$=\int\left(2\sin x\cos x\right)\left(\frac{1}{\sin x}\right)dx+C$$

$$=2\sin x+C$$

$$\Rightarrow y=2\sin^2 x+C\sin x,\ C\in\mathbb{R}$$

Example 13

Solve the differential equation $y' - y + y^4 = 0$, putting $z = \frac{1}{y^3}$.

Solution

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{-3}{y^4}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow -\frac{1}{3}y^4\frac{\mathrm{d}z}{\mathrm{d}x} - y + y^4 = 0, \quad y \neq 0$$

$$\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x} = 3 - 3z$$

$$\Rightarrow \frac{\mathrm{d}z}{z-1} = -3\mathrm{d}x$$

$$\Rightarrow \ln|z-1| = -3x + C$$

$$\Rightarrow z - 1 = \mathrm{k}e^{-3x}$$

$$\Rightarrow z = 1 + \mathrm{k}e^{-3x}$$

$$\Rightarrow y = \sqrt[3]{\frac{1}{1+\mathrm{k}e^{-3x}}}, \ \mathrm{k} \in \mathbb{R}$$

Example 14

a) Use the substitution $u = \frac{1}{y^2}$, $(y > 0)$ to transform the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = xe^{-x^2}y^3$

(I) into the differential equation $\frac{\mathrm{d}u}{\mathrm{d}x} - 4xu = -2xe^{-x^2}$ (II)

b) By finding the general solution of (II) find the general solution of (I)

Solution

$$u = \frac{1}{y^2} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-2}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow -\frac{1}{2}y^3\frac{\mathrm{d}u}{\mathrm{d}x} + 2xy = xe^{-x^2}y^3$$

$$\Rightarrow \frac{du}{dx} - \frac{4x}{y^2} = -2xe^{-x^2}$$

$$\Rightarrow \frac{du}{dx} - 4xu = -2xe^{-x^2}$$

$$v(x) = e^{\int -4x dx} = e^{-2x^2}$$

$$u(x) = e^{2x^2} \int \left(e^{-2x^2}\right)\left(-2xe^{-x^2}\right) dx$$

$$= e^{2x^2} \int \left(e^{-2x^2}\right)\left(-2xe^{-x^2}\right) dx = e^{2x^2} \int -2xe^{-3x^2} dx$$

$$u = e^{2x^2}\left[\frac{1}{3}e^{-3x^2} + C\right] = \frac{1}{3}e^{-x^2} + Ce^{2x^2}$$

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + Ce^{2x^2} \Rightarrow y = \frac{1}{\sqrt{\frac{1}{3}e^{-x^2} + Ce^{2x^2}}}$$

Example 15

a) Find the general solution of the differential equation $\frac{dy}{dx} - \frac{y}{x} = x \tanh x$, giving your answer in the form $y = f(x)$.

b) Given that k is a positive constant, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = e^{-kx}.$$

Solution

$$\rho = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x} y = \int \tanh x dx + C$$

$$\frac{1}{x} y = \ln(\cosh x) + C$$

$$\Rightarrow y = x\ln(\cosh x) + Cx, \quad C \in \mathbb{R}$$

$$r^2 - 2kr + k^2 = 0 \Rightarrow r_1 = r_2 = k$$

$$\text{Let } y_p = Ce^{-kx} \Rightarrow k^2Ce^{-kx} + 2k^2Ce^{-kx} + k^2Ce^{-kx} = e^{-kx}$$

$$\Rightarrow 4k^2C = 1 \Rightarrow C = \frac{1}{4k^2}$$

$$\Rightarrow y = e^{kx}(A + Bx) + \frac{1}{4k^2}e^{-kx}$$

Example 16

a) Use the substitution $u = \frac{1}{y^2}$, $(y > 0)$ to transform the differential equation $\frac{dy}{dx} + 2xy = xe^{-x^2}y^3$ (I) into the differential equation $\frac{du}{dx} - 4xu = -2xe^{-x^2}$ (II)

b) By finding the general solution of (II) find the general solution of (I)

Solution

$$u = \frac{1}{y^2} \Rightarrow \frac{du}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$$

$$\Rightarrow -\frac{1}{2}y^3\frac{du}{dx} + 2xy = xe^{-x^2}y^3$$

$$\Rightarrow \frac{du}{dx} - \frac{4x}{y^2} = -2xe^{-x^2}$$

$$\Rightarrow \frac{du}{dx} - 4xu = -2xe^{-x^2}$$

$$v(x) = e^{\int -4x\,dx} = e^{-2x^2}$$

$$u(x) = e^{2x^2}\int\left(e^{-2x^2}\right)\left(-2xe^{-x^2}\right)dx$$

$$= e^{2x^2}\int\left(e^{-2x^2}\right)\left(-2xe^{-x^2}\right)dx = e^{2x^2}\int -2xe^{-3x^2}\,dx$$

$$u = e^{2x^2}\left[\frac{1}{3}e^{-3x^2} + C\right] = \frac{1}{3}e^{-x^2} + Ce^{2x^2}$$

$$\frac{1}{y^2}=\frac{1}{3}e^{-x^2}+Ce^{2x^2} \Rightarrow y=\frac{1}{\sqrt{\frac{1}{3}e^{-x^2}+Ce^{2x^2}}}$$

Example 17

Find the general solution of the differential equation $\cos x\frac{dy}{dx}+y\sin x=\sin x\cos^3 x$,

$0<x<\frac{\pi}{2}$.

Solution

$$\cos x\frac{dy}{dx}+y\sin x=\sin x\cos^3 x$$
$$\frac{dy}{dx}+(\tan x)y=\sin x\cos^2 x$$

$$v(x)=e^{\int \tan x dx}=e^{\ln\sec x}=\sec x$$

$$y=\cos x\int \sin x\cos x dx=\cos x\left[\frac{\sin^2 x}{2}+C\right]$$

$$=\frac{1}{2}\cos x\sin^2 x+C\cos x$$

Example 18

a) Find $\int\frac{dx}{\sqrt{9x^2+4}}$

b) Find the general solution of the differential equation $(9x^2+4)\frac{dy}{dx}+9xy=1$

Solution

$$\int\frac{dx}{\sqrt{9x^2+4}}=\frac{1}{3}\sinh^{-1}\left(\frac{3x}{2}\right)+C$$

$$\frac{dy}{dx}+\frac{9x}{9x^2+4}y=\frac{1}{9x^2+4}$$

$$\Rightarrow \rho=\sqrt{9x^2+4}$$

$$\sqrt{9x^2+4}\,y=\int\sqrt{9x^2+4}\,\frac{\mathrm{d}x}{9x^2+4}$$

$$\Rightarrow y=\frac{1}{\sqrt{9x^2+4}}\left[\frac{1}{3}\sinh^{-1}\left(\frac{3x}{2}\right)+C\right]$$

Example 19

Given that $y = 1$ at $x = 0$, solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}-y\tan x=\sec^3 x$, giving your solution in the form $y = f(x)$.

Solution

$$\rho=\mathrm{e}^{-\tan x}=\cos x$$

$$\Rightarrow(\cos x)\,y=\int\sec^2 x\,\mathrm{d}x+C=\tan x+C$$

$$\Rightarrow y=\sec x\tan x+C\sec x,\quad C\in\mathbb{R}$$

Example 20

Use the substitution $y = vx$, where v is a function of x, to transform the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+xy+y^2$, $x > 0$ into the differential equation $x\frac{\mathrm{d}v}{\mathrm{d}x}=1+v^2$, $x > 0$. Hence, given that $y=\sqrt{3}$ at $x = 1$, find y in terms of x.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}v}{\mathrm{d}x}+v$$

$$\Rightarrow x^3\frac{\mathrm{d}v}{\mathrm{d}x}+x^2v=x^2+x^2v+v^2x^2$$

$$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x}=1+v^2$$

$$\Rightarrow\int\frac{\mathrm{d}v}{1+v^2}=\int\frac{\mathrm{d}x}{x}\Rightarrow\tan^{-1}v=\ln x+C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \ln x + C$$

$$y = \sqrt{3} \text{ at } x = 1 \Rightarrow \tan^{-1}\sqrt{3} = C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \ln x + \frac{\pi}{3} \quad \text{or} \quad y = x\tan\left(\ln x + \frac{\pi}{3}\right)$$

Example 21

$$\frac{dy}{dx} + y\left(1 + \frac{3}{x}\right) = \frac{1}{x^2}, \; x > 0$$

a) Verify that $x^3 e^x$ is an integrating factor for the differential equation.

b) Find the general solution of the differential equation.

Solution

$$\rho = e^{\int\left(1+\frac{3}{x}\right)dx} = e^{x+3\ln x} = e^x e^{\ln x^3} = x^3 e^x$$

$$x^3 e^x y = \int\left(\frac{1}{x^2}\right)\left(x^3 e^x\right)dx + C$$

$$x^3 e^x y = xe^x - e^x + C$$

$$y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{C}{x^3 e^x}$$

Example 22

Use the substitution $u = \ln y$ to solve the differential equation

$$\frac{1}{y}\cdot\frac{dy}{dx} + \frac{1}{x}\ln y = 1 \; (x > 0)$$

Solution

$$\frac{du}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$$

By susbstitution $\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{1}{x}\cdot u=1$

$$v(x)=e^{\int \frac{1}{x}dx}=e^{\ln x}=x$$

$$u(x)=\frac{1}{x}\int x\,dx=\frac{1}{x}\left[\frac{x^2}{2}+C\right]=\frac{x}{2}+\frac{C}{x}$$

$$y(x)=e^u=e^{\frac{x}{2}+\frac{C}{x}}$$

Example 23

Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}+2y\cot 2x=\sin x$, $0<x<\frac{\pi}{2}$, giving your answer in the form $y=f(x)$.

Solution

$$V(x)=e^{\int 2\cot 2x\,dx}=e^{\ln\sin 2x}=\sin 2x$$

$$y=\frac{1}{v(x)}\int V(x)Q(x)\,dx$$

$$=\csc 2x\int \sin 2x\sin x\,dx$$

$$=\csc 2x\left[\frac{2}{3}\sin^3 x+C\right]$$

Example 24

Find the particular solution of the given equation $\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x+1}=x$, $x>0$ and $y=2$ at $x=1$.

Solution

$$\rho=e^{-\int\frac{1}{x+1}dx}=e^{-\ln(x+1)}=\frac{1}{x+1}$$

$$y\left(\frac{1}{x+1}\right)=\int \frac{x}{x+1}\mathrm{d}x+\mathrm{C}$$

$$y\left(\frac{1}{x+1}\right)=\int\left(1-\frac{1}{x+1}\right)\mathrm{d}x+\mathrm{C}$$

$$y\left(\frac{1}{x+1}\right)=x-\ln(x+1)+\mathrm{C}$$

$$y=x(x+1)-(x+1)\ln(x+1)+\mathrm{C}(x+1)$$

$$y=2 \text{ at } x=1 \Rightarrow 2=2-2\ln 2+2\mathrm{C}$$

$$\Rightarrow \mathrm{C}=\ln 2$$

$$\Rightarrow y=(x+1)\left(x-\ln(x+1)+\ln 2\right)$$

$$=(x+1)\left(x+\ln\frac{2}{x+1}\right)$$

Example 25

Solve $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=x+xy$, $y=1$ at $x=0$.

<u>Solution</u>

$$\int \frac{\mathrm{d}y}{y(1+y)}=\int x\mathrm{d}x$$

$$\int\left(\frac{1}{y}-\frac{1}{1+y}\right)\mathrm{d}y=\int x\mathrm{d}x$$

$$\Rightarrow \ln|y|-\ln|1+y|=\frac{x^2}{2}+\mathrm{C}$$

$$\Rightarrow \ln\left|\frac{y}{1+y}\right|=\frac{x^2}{2}+\mathrm{C}, y=1 \text{ at } x=0$$

$$\Rightarrow \mathrm{C}=\ln\frac{1}{2}$$

$$\Rightarrow \frac{2y}{1+y}=\mathrm{e}^{x^2/2}$$

$$\Rightarrow y=\frac{1}{2-\mathrm{e}^{x^2/2}}$$

Example 26

Find the general solution of the differential equation $\frac{dy}{dx} + y(\tanh x) = \frac{\operatorname{sech} x}{x}, x > 0$

Solution

$$V(x) = \mathrm{e}\ \mathbf{V(x)=e^{\int \tanh x\,dx} = \cosh x}$$

$$\boldsymbol{y = \operatorname{sech} x \int \frac{1}{x} dx = \operatorname{sech} x (\ln x + C)}$$

Example 27

Solve each of the following differential equations.

a) $y'' - 5y' - 6y = 0$

b) $y'' + 4y' + 4y = 0$

c) $y'' - 2y' + 2y = 0$

Solution

$$\mathbf{r^2 - 5r - 6 = 0 \Rightarrow r = 6, -1}$$

$$\boldsymbol{\Rightarrow y = C_1 e^{6x} + C_2 e^{-x}, \quad C_1, C_2 \in \mathbb{R}}$$

$$\mathbf{r^2 + 4r + 4 = 0 \Rightarrow r_1 = r_2 = -2}$$

$$\boldsymbol{\Rightarrow y = e^{-2x}(C_1 + C_2 x), \quad C_1, C_2 \in \mathbb{R}}$$

$$\mathbf{r^2 - 2r + 2 = 0 \Rightarrow r = 1 \mp \sqrt{-1} = 1 \mp i}$$

$$\boldsymbol{\Rightarrow y = e^{x}(C_1 \sin x + C_2 \cos x), \quad C_1, C_2 \in \mathbb{R}}$$

Example 28

Given that y = 1 at x = 0, solve the differential equation $\frac{dy}{dx} - y \tan x = \sec^3 x$, giving your answer in the form $y = f(x)$.

<u>Solution</u>

$$\int -\tan x\,dx = \ln|\cos x|$$

$$\Rightarrow \rho = \cos x$$

$$\cos x y = \int \cos x \frac{1}{\cos^3 x}dx + C$$

$$y = \sec x(\tan x + C)$$

$$y = 1 \text{ at } x = 0 \Rightarrow y = \sec x \tan x + \sec x$$

Example 29

Find the general solution of the differential equation $\frac{dy}{dx} - \frac{2}{x}y = x^2 \ln x$, $x > 0$.

<u>Solution</u>

$$\rho = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{1}{x^2}y = \int \left(x^2 \ln x\right)\left(\frac{1}{x^2}\right)dx + c$$

$$= \int \ln x\,dx + C = x\ln x - x + c$$

$$\Rightarrow y = x^3 \ln x - x^3 + cx^2 \ ,\ c \in \mathbb{R}$$

Example 30

Solve the following differential equation: $\sin x \frac{dy}{dx} = \tan y\left(3\cos x + \sin x\right)$, $0 < x \le \frac{\pi}{2}$

and $0 < y < \frac{\pi}{2}$ knowing that $y = \frac{\pi}{6}$ at $x = \frac{\pi}{2}$. Give your answer in the form

$\sin y = f(x)$.

<u>Solution</u>

$$\sin x\frac{dy}{dx}=\tan y\left(3\cos x+\sin x\right)$$

$$\cot y dy=\frac{\left(3\cos x+\sin x\right)}{\sin x}dx$$

$$\ln\left(\sin y\right)=3\ln\left(\sin x\right)+x+C$$

$$\ln\frac{1}{2}=3\ln 1+C\Rightarrow C=-\ln 2$$

$$\ln\left(\sin y\right)=\ln\left(\frac{\sin^3 x}{2}\right)+x$$

$$\sin y=\frac{\sin^3 x}{2}\cdot e^x$$

Example 31

a) Use the substitution $y=vx$, where v is a function of x to transform the differential equation $\frac{dy}{dx}=1+\frac{y}{x}+\left(\frac{y}{x}\right)^2$, $x>0$ into a differential equation in v and x.

b) Using par(a), find y in terms of x given that $y=1$ at $x=1$

Solution

$$y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\frac{dy}{dx}=1+\frac{y}{x}+\left(\frac{y}{x}\right)^2\Rightarrow v+x\frac{dv}{dx}=1+v+v^2$$

$$\Rightarrow x\frac{dv}{dx}=1+v^2$$

$$\frac{dv}{1+v^2}=\frac{dx}{x}\Rightarrow\tan^{-1}v=\ln x+C$$

$$\Rightarrow\tan^{-1}\frac{y}{x}=\ln x+C$$

$$\tan^{-1}1=C\Rightarrow C=\frac{\pi}{4}$$

$$\tan^{-1}\frac{y}{x}=\ln x+\frac{\pi}{4}$$

$$\frac{y}{x}=\tan\left(\ln x+\frac{\pi}{4}\right)$$

$$y=x\tan\left(\ln x+\frac{\pi}{4}\right)$$

Example 32

Find the solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}+y\cot x=\sin 2x,\quad 0<x<\pi$ for which $y=1$ at $x=\frac{\pi}{4}$.

Solution

$$\rho=\mathrm{e}^{\int\cot x\mathrm{d}x}=\sin x$$

$$\Rightarrow(\sin x)y=\int\sin 2x\sin x\mathrm{d}x+\mathrm{C}$$

$$=2\int\sin^2 x\cos x\mathrm{d}x+\mathrm{C}$$

$$=\frac{2}{3}\sin^3 x+\mathrm{C}$$

$$y=\frac{2}{3}\sin^2 x+\mathrm{C}\csc x,\quad \mathrm{C}\in\mathbb{R}$$

$$y=\frac{\cosh^{-1}x}{\sqrt{x^2-1}}+\frac{\mathrm{C}}{\sqrt{x^2-1}},\quad \mathrm{C}\in\mathbb{R}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}u}{\mathrm{d}x}x-u}{x^2}=\frac{1}{x}\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{u}{x^2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\left(\mathrm{d}^2u/\mathrm{d}x^2\right)x-\mathrm{d}u/\mathrm{d}x}{x^2}-\frac{x^2\,\mathrm{d}u/\mathrm{d}x-2xu}{x^4}$$

$$=\frac{1}{x}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\frac{1}{x^2}\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{1}{x^2}\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{2u}{x^3}$$

$$\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{x}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\frac{2}{x^2}\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{2u}{x^3}$$

$$\frac{1}{x}\frac{d^2u}{dx^2}-\frac{2}{x^2}\frac{du}{dx}+\frac{2u}{x^3}+\frac{2}{x}\left(\frac{1}{x}\frac{du}{dx}-\frac{u}{x^2}\right)+25\left(\frac{u}{x}\right)=0$$

$$\Rightarrow \frac{d^2u}{dx^2}+25u=0$$

$$\Rightarrow u=A\cos 5x+B\sin 5x\Rightarrow y=\frac{1}{x}\left(A\cos 5x+B\sin 5x\right)$$

Example 33

i) Use the substitution $y=vx,$ where v is a function of x, to transform the differential equation

$x^2\frac{dy}{dx}=x^2+xy+y^2,\ x>0$ into the differential equation $x\frac{dv}{dx}=1+v^2,\ x>0.$

ii) Hence, given $y=\sqrt{3}$ at $x = 1$, find y in terms of x.

$$y=vx\Rightarrow y'=v'x+v$$

$$\Rightarrow x^2\frac{dy}{dx}=x^2\left(x\frac{dv}{dx}+v\right)=x^3\frac{dv}{dx}+x^2v=$$
$$x^2+x^2v+v^2x^2$$

$$\Rightarrow x^3\frac{dv}{dx}=x^2+x^2v^2\Rightarrow x\frac{dv}{dx}=1+v^2$$

$$\frac{dv}{1+v^2}=\frac{1}{x}dx$$

$$\Rightarrow \tan^{-1}v=\ln x+c\quad (\text{since } x>0)$$

$$\Rightarrow v=\tan(\ln x+c)$$

$$\Rightarrow y=x\tan(\ln x+c)$$

$$x=1,\ \ y=\sqrt{3}\Rightarrow \sqrt{3}=\tan(c)\Rightarrow c=\frac{\pi}{3}$$
$$\Rightarrow y=x\tan\left(\ln x+\frac{\pi}{3}\right)$$

Example 34

$$\frac{dy}{dx} + y\left(1 + \frac{3}{x}\right) = \frac{1}{x^2}, \ x > 0$$

a) Verify that x^3e^x is an integrating factor for the differential equation.

b) Find the general solution of the differential equation.

Solution

$$\boldsymbol{\rho = e^{\int\left(1+\frac{3}{x}\right)dx} = e^{x+3\ln x} = e^x e^{\ln x^3} = x^3 e^x}$$

$$\boldsymbol{x^3e^x y = \int\left(\frac{1}{x^2}\right)\left(x^3e^x\right)dx + C}$$

$$\boldsymbol{x^3e^x y = xe^x - e^x + C}$$

$$\boldsymbol{y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{C}{x^3e^x}}$$

Example 35

Find the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$

Solution

$$\mathbf{r^2 + 2r - 3 = 0}$$

$$\boldsymbol{\Rightarrow r = -3, 1}$$

$$\boldsymbol{\Rightarrow y_h = C_1e^x + C_2e^{-3x}}$$

$$\boldsymbol{y_p = -2}$$

$$\boldsymbol{\Rightarrow y = C_1e^x + C_2e^{-3x} - 2}$$

Section: 2. Second Order Linear Differential Equations

Example 1

a) Show that the transformation $y = vx$ transforms the equation :

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(2+9x^2\right)y = x^5, \quad \text{I}$$

into the equation : $\dfrac{d^2v}{dx^2} + 9v = x^2, \quad \text{II}$

b) Solve the differential equation II to find v as a function of x.

c) Hence state the general solution of the differential equation I.

<u>Solution</u>

$$\frac{dy}{dx} = v + x\frac{dv}{dx}, \quad \frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

$$\Rightarrow x^2\left(2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right) - 2x\left(v + x\frac{dv}{dx}\right) + \left(2+9x^2\right)(vx) = x^5$$

$$\Rightarrow 9x^3v + x^3\frac{d^2v}{dx^2} = x^5$$

$$\Rightarrow 9v + \frac{d^2v}{dx^2} = x^2$$

$$r^2 + 9 = 0 \Rightarrow r = \mp 3i$$

$$\Rightarrow v_h = A\cos 3x + B\sin 3x$$

$$v_p = Cx^2 + Dx + E$$

$$\Rightarrow C = \frac{1}{9}, \quad E = \frac{-2}{81} \text{ and } D = 0$$

$$\Rightarrow v = A\cos 3x + B\sin 3x + \frac{1}{9}x^2 - \frac{2}{81}$$

$$y = \frac{2}{81}x + \frac{1}{9}x^3 + x\left(A\cos 3x + B\sin 3x\right)$$

Example 2

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 4 - 2x^2$

Solution

$$\frac{d^2y}{dx^2} - y = 4 - 2x^2$$

$$r^2 - 1 = 0 \Rightarrow r = \pm 1 \Rightarrow y = A\,e^{x} + Be^{-x}$$

$$y_P = Mx^2 + Nx + P$$

$$y_P' = 2Mx + N$$
$$y_P'' = 2M$$

$$2M - Mx^2 - Nx - P = 4 - 2x^2$$
$$\Rightarrow N = 0,\ M = 2 \text{ and } P = 0$$

$$\Rightarrow y_G = Ae^{x} + Be^{-x} + 2x^2$$

Example 3

Transform the equation $\frac{d^2y}{dx^2} + x^2 + y + 2 = 0$ to $\frac{d^2t}{dx^2} + t = 0,$ by means of the substitution $y = t - x^2$, and hence find the general solution.

Solution

$$\frac{dy}{dx} = \frac{dt}{dx} - 2x$$

$$\frac{d^2y}{dx^2} = \frac{d^2t}{dx^2} - 2$$

$$\Rightarrow \frac{d^2t}{dx^2} - 2 + x^2 + t - x^2 + 2 = 0$$
$$\Rightarrow \frac{d^2t}{dx^2} + t = 0$$

$$\Rightarrow t = A\cos x + B\sin x$$

$$\Rightarrow y = A\cos x + B\sin x - x^2$$

Example 4

a) Solve $16y'' + y = 0$, $y(0) = 2$ and $y'(0) = -1$

b) Use the substitution $zy^2 = 1$ to transform the differential equation $\frac{dy}{dx} + y = xy^3$ into an equation containing z and x. Solve this equation and hence find y in terms of x.

Solution

$$16y'' + y = 0 \quad y(0) = 2 \quad y'(0) = -1$$

$$16r^2 + 1 = 0 \quad r^2 = \frac{-1}{16} \quad r = \pm\frac{1}{4}i$$

$$y = A\cos\frac{1}{4}x + B\sin\frac{1}{4}x$$

$$y(0) = 2 \Rightarrow A = 2$$

$$y' = \frac{-A}{4}\sin\frac{x}{4} + \frac{B}{4}\cos\frac{x}{4}$$

$$y'(0) = \frac{B}{4} = -1 \Rightarrow B = -4$$

$$\Rightarrow y = 2\cos\frac{x}{4} - 4\sin\frac{x}{4}$$

$$\frac{dy}{dx} + y = xy^3 \quad \text{and} \quad z = \frac{1}{y^2}$$

$$\frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$

$$\Rightarrow \frac{-y^3}{2}\frac{dz}{dx} + y = xy^3 \Rightarrow \frac{dz}{dx} - 2z = -2x$$

$$z = \frac{1}{e^{-2x}}\int(-2x)e^{-2x}dx$$

$$\begin{array}{c|l} 2x & e^{-2x} \\ 2 & -1/2\,e^{-2x} \\ 0 & 1/4\,e^{-2x} \end{array}$$

$$\Rightarrow z = \frac{-1}{e^{-2x}}\left[-xe^{-2x} - \frac{1}{2}e^{-2x} + C\right]$$

$$\Rightarrow z = x + \frac{1}{2} + Ce^{2x}$$

$$\Rightarrow y^2 = \frac{1}{x + 0.5 + Ce^{2x}} \Rightarrow y = \pm\sqrt{\frac{1}{x + 0.5 + Ce^{2x}}}$$

Example 5

Solve the differential equation $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 2x^2$ using the substitution $x = e^t$.

Solution

$$x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 2x^2$$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right)e^{-t} = e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$$

$$\Rightarrow \left(e^{2t}\right)\left(e^{-2t}\right)\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 4\frac{dy}{dt} + 6y = 2e^{2t}$$

$$\Rightarrow \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 2e^{2t}$$

$$\Rightarrow y_H = Ae^{2t} + Be^{3t}$$

$$y_p = Mte^{2t}$$

$$y_{p'} = 2Mt^{2t} + Me^{2t}$$

$$y_{p''} = 4Me^{2t} + 4Mte^{2t}$$

$$\Rightarrow \mathbf{4Me^{2\mathit{t}} + 4M\mathit{t}e^{2\mathit{t}} - 10M\mathit{t}e^{2\mathit{t}} - 5Me^{2\mathit{t}} + 6M\mathit{t}e^{2\mathit{t}} = 2e^{2\mathit{t}}}$$

$$\Rightarrow \mathbf{(4M\mathit{t} - 10M\mathit{t} + 6M\mathit{t})e^{2\mathit{t}} - Me^{2\mathit{t}} = 2e^{2\mathit{t}}}$$

$$\Rightarrow \mathbf{M = -2}$$

$$\Rightarrow \boldsymbol{y_G = Ae^{2t} + Be^{3t} - 2te^{2t}}$$

$$\boldsymbol{y = Ae^{2\ln x} + Be^{3\ln x} - 2\ln x e^{2\ln x}}$$
$$\boldsymbol{y = Ax^2 + Bx^3 - 2x^2 \ln x}$$

Example 6

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t}, \quad t \geq 0$$

a) Show that Kt^2e^{3t} is a particular integral of the differential equation, where K is a constant to be found.

b) Find the general solution of the differential equation.

Given that a particular solution satisfies $y = 3$ and $\frac{dy}{dt} = 1$ when $t = 0$,

c) Find this solution.

Solution

$$\boldsymbol{y' = 2Kt\, e^{3t} + 3Kt^2e^{3t}}$$

$$\boldsymbol{y'' = 2Ke^{3t} + 6Kte^{3t} + 6Kte^{3t} + 9Kt^2e^{3t}}$$
Substitute you get $\mathbf{K = 2}$

$$\boldsymbol{r^2 - 6r + 9 = 0 \Rightarrow r = 3 \Rightarrow y_h = e^{3t}(A + Bt)}$$

$$\Rightarrow \boldsymbol{y = (A + Bt)e^{3t} + 2t^2e^{3t}}$$

$$\boldsymbol{A = 3 \text{ and } B = -8 \Rightarrow y = (3 - 8t + 2t^2)e^{3t}}$$

Example 7

a) Given that $y = e^x \sin x$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

b) Show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$

c) By differentiating the equation in part (b), or otherwise, obtain the Maclaurin expansion of $\mathrm{e}^x \sin x$ up to and including the term in x^3.

Solution

$$\boldsymbol{y' = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x = \mathrm{e}^x (\sin x + \cos x)}$$

$$\boldsymbol{y'' = \mathrm{e}^x (\sin x + \cos x) + \mathrm{e}^x (\cos x - \sin x) = 2\mathrm{e}^x \cos x}$$

$$\boldsymbol{2\mathrm{e}^x \cos x - 2\left(\mathrm{e}^x\right)(\sin x + \cos x) + 2\mathrm{e}^x \sin x = y'' - 2y' + 2y}$$

$$\boldsymbol{= 2\mathrm{e}^x \cos x - 2\mathrm{e}^x \sin x - 2\mathrm{e}^x \cos x + 2\mathrm{e}^x \sin x = 0}$$

$$\boldsymbol{y''' - 2y'' + 2y' = 0 \Rightarrow y''' = 2y'' - 2y'}$$

$$\boldsymbol{y(0) = \mathrm{e}^0 \sin 0 = 0}$$
$$\boldsymbol{y'(0) = 1}$$
$$\boldsymbol{y''(0) = 2}$$

$$\boldsymbol{y \approx y(0) + y'(0)\,x + \frac{y''(0)\,x^2}{2!} + \frac{y'''(0)\,x^3}{3!}}$$

$$\boldsymbol{y'''(0) = 2(2) - 2 = 2}$$

$$\boldsymbol{\Rightarrow y \approx x + x^2 + \frac{x^3}{3}}$$

Example 8

A particle moves along the x-axis so that its position function $x(t)$ satisfies the differential equation $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \frac{\mathrm{d}x}{\mathrm{d}t} - 6x = 0$ and has the property that at time $t = 0$, $x = 2$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -9$.

a) Write an expression for $x(t)$ in terms of *t*.

b) At what time *t*, if any, does the particle pass through the origin?

c) At what time *t*, if any, is the particle is at rest?

<u>Solution</u>

$$\mathbf{r^2 - r - 6 = 0 \Rightarrow (r-3)(r+2) = 0 \Rightarrow r = 3 \text{ or } r = -2}$$

$$\Rightarrow \boldsymbol{x = \mathrm{A}e^{3t} + \mathrm{B}e^{-2t}}$$

$$\left.\begin{array}{l} \boldsymbol{t = 0,\ x = 2 \Rightarrow 2 = \mathrm{A} + \mathrm{B}} \\ \boldsymbol{t = 0,\ x' = -9 \Rightarrow 3\mathrm{A} - 2\mathrm{B} = -9} \end{array}\right\} \Rightarrow \mathbf{A = -1,\ B = 3}$$

$$\Rightarrow \boldsymbol{x = 3\mathrm{e}^{-2t} - \mathrm{e}^{3t}}$$

Particle passes through the origin $\Rightarrow \boldsymbol{x = 0}$

$$\Rightarrow \boldsymbol{3\mathrm{e}^{-2t} = \mathrm{e}^{3t} \Rightarrow \mathrm{e}^{5t} = 3 \Rightarrow 5t = \ln 3}$$

$$\Rightarrow \boldsymbol{t = \frac{\ln 3}{5}}$$

Particle is at rest $\Rightarrow \boldsymbol{x' = 0 \Rightarrow 3\mathrm{A}\mathrm{e}^{3t} - 2\mathrm{B}\mathrm{e}^{-2t} = 0}$

$$\Rightarrow \boldsymbol{-3\mathrm{e}^{3t} - 6\mathrm{e}^{-2t} = 0 \Rightarrow t \in \phi}$$

Example 9

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 65\sin 2x, \quad x > 0$$

Find the general solution of the differential equation.

<u>Solution</u>

$$\mathbf{r^2 + 4r + 5 = 0}$$
$$\mathbf{\Rightarrow r = -2 \mp \sqrt{-1} = -2 \mp i}$$
$$\Rightarrow \boldsymbol{y_{\mathbf{h}} = \mathrm{e}^{-2x}\left(\mathrm{M}\cos x + \mathrm{N}\sin x\right)}$$

$$y_p = A\sin 2x + B\cos 2x$$
$$y_p' = 2A\cos 2x - 2B\sin 2x$$

$$y_p'' = -4A\sin 2x - 4B\cos 2x$$

$$\left.\begin{array}{l} -4A - 8B + 5A = 65 \\ -4B + 8A + 5B = 0 \end{array}\right\} \Rightarrow A = 1,\ B = -8$$

$$\Rightarrow y = \sin 2x - 8\cos 2x + e^{-2x}\left(M\cos x + N\sin x\right)$$

Example 10

Given $x = At^2e^{-t}$ satisfies the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t}$

a) Find the value of A.

b) Hence find the solution of the differential equation for which $x = 1$ and $\frac{dx}{dt} = 0$ at $t = 0$.

c) Use your solution to prove that for $t \geq 0,\ x \leq 1$.

Solution

$$x = At^2e^{-t} \Rightarrow x' = 2Ate^{-t} - At^2e^{-t} = e^{-t}\left[2At - At^2\right]$$

$$x'' = -e^{-t}\left[2At - At^2\right] + e^{-t}\left[2A - 2At\right]$$
$$= e^{-t}\left[2A - 2At - 2At + At^2\right] = e^{-t}\left[At^2 - 4At + 2A\right]$$

$$\Rightarrow e^{-t}\left[At^2 - 4At + 2A\right] + 2e^{-t}\left[2At - At^2\right] + At^2e^{-t} = e^{-t}$$
$$\Rightarrow 2Ae^{-t} = e^{-t} \Rightarrow A = 1/2$$

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1 \Rightarrow x_h = Ae^{-t} + Bte^{-t}$$

$$\Rightarrow x_g = Ae^{-t} + Bte^{-t} + 1/2\,t^2e^{-t} = e^{-t}\left[A + Bt + 1/2\,t^2\right]$$

$$x(0) = 1 \Rightarrow A = 1$$

$$x' = -e^{-t}\left[A + Bt + 1/2\,t^2\right] + e^{-t}\left[B + t\right]$$

$$x'(0) = 0 \Rightarrow -A + B \Rightarrow A = B = 1$$
$$\Rightarrow x = e^{-t}\left[1 + t + 1/2t^2\right]$$

$$x' = e^{-t}\left[t + 1 - 1 - t - 1/2t^2\right] = -e^{-t}\left[1/2t^2\right]$$

$x' < 0 \quad \forall t \neq 0$ **for** $t \geq 0$ **the function is decreasing** $\Rightarrow$ **maximum value occurs at**

$t = 0 \qquad x(0) = 1$ **is absolute maximum on** $[0, \infty)$

Example 11

By using the substitution $x = \ln t, t > 0$, transform the differential equation

$\dfrac{d^2y}{dx^2} + \left(8e^x - 1\right)\dfrac{dy}{dx} + 17ye^{2x} = 3e^{2x}\cos 2e^x$ into a differential equation for y in terms of t and hence solve for y in terms of x.

Solution

$$\frac{d^2y}{dx^2} + \left(8e^x - 1\right)\frac{dy}{dx} + 17ye^{2x} = 3e^{2x}\cos 2x$$

$$x = \ln t \Rightarrow \frac{dx}{dt} = \frac{1}{t} \Rightarrow \frac{dt}{dx} = t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = t\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right)(t) = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

$$\Rightarrow t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 8t^2\frac{dy}{dt} - t\frac{dy}{dt} + 17yt^2 = 3t^2\cos 2t$$

$$\Rightarrow \frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 17y = 3\cos 2t$$

$$\Delta = 64 - (4)(17) = 64 - 68 = -4$$
$$\Rightarrow r = \frac{-8 \mp 2i}{2} = -4 \mp i$$

$$\Rightarrow y_H = e^{-4t}\left[A\cos t + B\sin t\right]$$

$$y_{\mathrm{p}} = \mathrm{M}\cos 2t + \mathrm{N}\sin 2t$$

$$y_{\mathrm{p}'} = 2\mathrm{N}\cos 2t - 2\mathrm{M}\sin 2t$$

$$y_{\mathrm{p}''} = -4\mathrm{N}\sin 2t - 4\mathrm{M}\cos 2t$$

$$-4\mathrm{N}\sin 2t - 4\mathrm{M}\cos 2t + 8\left(2\mathrm{N}\cos 2t - 2\mathrm{M}\sin 2t\right) + 17\left(\mathrm{MC} + \mathrm{NS}\right) = 3\mathrm{C}$$

$$\Rightarrow -4\mathrm{NS} - 4\mathrm{MC} + 16\mathrm{NC} - 16\mathrm{MS} + 17\mathrm{MC} + 17\mathrm{NS} = 3\mathrm{C}$$

$$\Rightarrow 13\mathrm{N}\sin 2t - 16\mathrm{M}\sin 2t + \left(13\mathrm{M} + 16\mathrm{N}\right)\cos 2t = 3\cos 2t$$

$$\Rightarrow \left.\begin{matrix} 13\mathrm{N} - 16\mathrm{M} = 0 \\ 13\mathrm{M} + 16\mathrm{N} = 3 \end{matrix}\right\} \Rightarrow \mathrm{M} = \frac{39}{425},\ \mathrm{N} = \frac{48}{425}$$

Note: C stands for cos2t and s for sin2t

$$\Rightarrow y = \mathrm{e}^{-4t}\left[\mathrm{A}\cos t + \mathrm{B}\sin t\right] + \frac{39}{425}\cos 2t + \frac{48}{425}\sin 2t$$

$$\Rightarrow y = \mathrm{e}^{-4\mathrm{e}x}\left[\mathrm{A}\cos \mathrm{e}^{x} + \mathrm{B}\sin \mathrm{e}^{x}\right] + \frac{39}{425}\cos 2\mathrm{e}^{x} + \frac{48}{425}\sin 2e^{x}$$

Example 12

Find the general solution for $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 3\mathrm{e}^{2x}$.

<u>Solution</u>

$$\mathrm{r}^2 + 3\mathrm{r} + 2 = 0 \Rightarrow \mathrm{r} = -2, -1$$

$$\Rightarrow y_{\mathrm{h}} = \mathrm{A}\mathrm{e}^{-2x} + \mathrm{B}\mathrm{e}^{-x}\ ,\mathrm{A},\mathrm{B} \in \mathbb{R}$$

$$y_{\mathrm{p}} = \mathrm{C}\mathrm{e}^{2x}$$

$$\Rightarrow 4\mathrm{C}\mathrm{e}^{2x} + 6\mathrm{C}\mathrm{e}^{2x} + 2\mathrm{C}\mathrm{e}^{2x} \equiv 3\mathrm{e}^{2x}$$

$$\Rightarrow \mathrm{C} = \frac{1}{4}$$

$$\Rightarrow y_{\mathrm{p}} = \frac{1}{4}\mathrm{e}^{2x}$$

$$\Rightarrow y = \mathrm{A}\mathrm{e}^{-2x} + \mathrm{B}\mathrm{e}^{-x} + \frac{1}{4}\mathrm{e}^{2x}$$

Example 13

Given $x = \mathrm{e}^u$, where u is a function of x, show that

a) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u}$

b) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}^2y}{\mathrm{d}u^2} - \dfrac{\mathrm{d}y}{\mathrm{d}u}$ Hence find the general solution of the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}u} = x\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} = \left(\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\cdot\frac{\mathrm{d}x}{\mathrm{d}u}$$

$$= \left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}u} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\cdot\mathrm{e}^u$$

$$= \left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}u} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\cdot x$$

$$= \frac{\mathrm{d}y}{\mathrm{d}u} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

substitute

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{\mathrm{d}y}{\mathrm{d}u} - 5\frac{\mathrm{d}y}{\mathrm{d}u} + 9y = 0$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}u^2} - 6\frac{\mathrm{d}y}{\mathrm{d}u} + 9y = 0$$

$$\mathrm{r}^2 - 6\mathrm{r} + 9 = 0$$

$$\Rightarrow \mathrm{r} = 3$$

$$\Rightarrow y = (\mathrm{A} + \mathrm{B}u)\mathrm{e}^{3u}$$

$$\Rightarrow y = (\mathrm{A} + \mathrm{B}\ln x)x^3 \quad , \quad \mathrm{A,B} \in \mathbb{R}$$

Example 14

Find the general solution of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x + 3$

Solution

$$\mathbf{r^2 + r = 0 \Rightarrow r = 0,\ \ -1}$$

$$\Rightarrow \boldsymbol{y_h = C_1 + C_2 e^{-x}} \text{ and } \boldsymbol{y_p = Ax^2 + Bx + C}$$

$$\Rightarrow \mathbf{2A + 2A}\boldsymbol{x}\mathbf{ + B \equiv 2}\boldsymbol{x}\mathbf{ + 3}$$
$$\Rightarrow \mathbf{A = 1 \text{ and } B = 1}$$

$$\Rightarrow \boldsymbol{y = C_1 + C_2 e^{-x} + x^2 + x \ \ ,\ \ C_1, C_2 \in \mathbb{R}}$$

Example 15

Transform the equation $\frac{d^2y}{dx^2} + x^2 + y + 2 = 0$ to $\frac{d^2t}{dx^2} + t = 0,$ by means of the substitution $y = t - x^2$, and hence find the general solution.

Solution

$$\boldsymbol{\frac{dy}{dx} = \frac{dt}{dx} - 2x}$$

$$\boldsymbol{\frac{d^2y}{dx^2} = \frac{d^2t}{dx^2} - 2}$$

$$\boldsymbol{\Rightarrow \frac{d^2t}{dx^2} - 2 + x^2 + t - x^2 + 2 = 0}$$
$$\boldsymbol{\Rightarrow \frac{d^2t}{dx^2} + t = 0}$$

$$\boldsymbol{\Rightarrow t = A\cos x + B\sin x}$$
$$\boldsymbol{\Rightarrow y = A\cos x + B\sin x - x^2}$$

Example 16

a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

b) Using your answer to part (a), find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+25y=3\cos 5x$$

Given that at $x=0,\ y=0$ and $\frac{dy}{dx}=5$

c) Find the particular solution of this differential equation, giving your solution in the form $y=f(x)$.

Solution

$$\frac{dy}{dx}=\lambda\sin 5x+5\lambda x\cos 5x$$

$$\frac{d^2y}{dx^2}=5\lambda\cos 5x+5\lambda\cos 5x-25\lambda x\sin 5x$$
$$=10\lambda\cos 5x-25\lambda x\sin 5x$$
$$=10\lambda\cos 5x-25\lambda x\sin 5x$$

$$10\lambda\cos 5x-25\lambda x\sin 5x+25\lambda x\sin 5x=3\cos 5x$$
$$\lambda=\frac{3}{10}$$

$$r^2+25=0\Rightarrow r=\pm 5i$$

$$y=C_1\cos 5x+C_2\sin 5x+\frac{3}{10}x\sin 5x$$
$$0=C_1$$
$$\frac{dy}{dx}=5C_2\cos 5x+\frac{3}{10}\sin 5x+\frac{3}{2}x\cos 5x$$
$$5=5C_2+0\Rightarrow C_2=1$$

$$y=\sin 5x+\frac{3}{10}x\sin 5x$$

Example 17

Given that $y=\frac{u}{x}$, show that $\frac{d^2y}{dx^2}=\frac{1}{x}\frac{d^2u}{dx^2}-\frac{2}{x^2}\frac{du}{dx}+\frac{2u}{x^3}$. Hence find the general solution of the differential equation $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+25y=0$, $x>0$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x\frac{\mathrm{d}u}{\mathrm{d}x}-u}{x^2}$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\left(x\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{\mathrm{d}u}{\mathrm{d}x}\right)x^2-2x\left(x\frac{\mathrm{d}u}{\mathrm{d}x}-u\right)}{x^4}$$

$$=\frac{x^3\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-2x^2\frac{\mathrm{d}u}{\mathrm{d}x}+2xu}{x^4}$$

$$=\frac{1}{x}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\frac{2}{x^2}\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{2u}{x^3}$$

Substitute in the given equation we get

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+25u=0$$

$$\Rightarrow u=\mathrm{A}\cos 5x+\mathrm{B}\sin 5x$$

$$\Rightarrow y=\frac{1}{x}\left(\mathrm{A}\cos x+\mathrm{B}\sin x\right),\ \mathrm{A},\ \mathrm{B}\in\mathbb{R}$$

Example 18

Transform the equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x^2+y+2=0$ by means of the substitution $y=t-x^2$, and hence find the general solution.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}t}{\mathrm{d}x}-2x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}^2t}{\mathrm{d}x^2}-2$$

$$\Rightarrow \frac{\mathrm{d}^2t}{\mathrm{d}x^2}-2+x^2+t-x^2+2=0$$

$$\Rightarrow \frac{\mathrm{d}^2t}{\mathrm{d}x^2}+t=0$$

$\Rightarrow \boldsymbol{t = \mathrm{A}\cos x + \mathrm{B}\sin x}$

$\Rightarrow \boldsymbol{y = \mathrm{A}\cos x + \mathrm{B}\sin x - x^2}$

Example 19

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = e^{-3x}$

Solution

$\mathbf{r^2 + 4r + 13 = 0 \Rightarrow r = -2 \mp \sqrt{4-13} = -2 \mp 3i}$

$\Rightarrow \boldsymbol{y_h = e^{-2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)}$

$\boldsymbol{y_p = Ae^{-3x}}$

$\Rightarrow \mathbf{9A - 12A + 13A = 1}$

$\Rightarrow \mathbf{A = \frac{1}{10}}$

$\Rightarrow \boldsymbol{y = e^{-2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + \frac{1}{10}e^{-3x}}$

Example 20

Given that $x = \ln t$, $t > 0$ and that y is a function of x,

a) find $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and t,

b) show that $\frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$.

c) Show that the substitution $x = \ln t$ transforms the differential equation

$\frac{d^2y}{dx^2} - (1-6e^x)\frac{dy}{dx} + 10ye^{2x} = 5e^{2x}\sin 2e^x$ (I) into the differential equation

$\frac{d^2y}{dt^2} + 6\frac{d\,y}{dt} + 10y = 5\sin 2t$ (II)

d) Hence find the general solution of (I), giving your answer in the form $y = f\left(x\right)$.

Solution

$$\frac{dy}{dx}=t\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=t\frac{d}{dt}\left(\frac{dy}{dx}\right)=t\left(\frac{dy}{dt}+t\frac{d^2y}{dt^2}\right)=t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}$$

$$t^2y''+ty'-(1-6t)(ty')+(10y)\left(t^2\right)=5t^2\sin(2t)$$

$$\Rightarrow y''+6y'+10y=5\sin(2t)$$

$$r^2+6r+10=0\Rightarrow r=-3\mp i$$

$$\Rightarrow y=e^{-3t}\left[A\cos t+B\sin t\right]$$

$$y_p=M\sin 2t+N\cos 2t$$

$$y_p'=2M\cos 2t-2N\sin 2t$$

$$y_p''=-4M\sin 2t-4N\cos 2t$$

Substituting into equation we get $M=\frac{1}{6},\quad N=-\frac{1}{3}$

$$\Rightarrow y=e^{-3t}\left[A\cos t+B\sin t\right]+\frac{1}{6}\sin 2t-\frac{1}{3}\cos 2t,\quad t=e^x$$

Example 21

$$\frac{d^2y}{dx^2}=e^x\left(2y\frac{dy}{dx}+y^2+1\right)$$

a) Show that $\frac{d^3y}{dx^3}=e^x\left(2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+ky\frac{dy}{dx}+y^2+1\right)$

where k is a constant to be found.

Given that $x=0$, $y=1$ and $\frac{dy}{dx}=2$

b) Find a series solution for y in ascending powers of x, up to and including the term in x^3.

Solution

$$\frac{d^2y}{dx^2} = e^x\left(2y\frac{dy}{dx} + y^2 + 1\right)$$

$$\frac{d^3y}{dx^3} = e^x\left(2y\frac{dy}{dx} + y^2 + 1\right) + e^x\left[2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} + 2y\frac{dy}{dx}\right]$$

$$= e^x\left(2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 4y\frac{dy}{dx} + y^2 + 1\right)$$

$$\Rightarrow k = 4$$

$$x = 0,\ y = 1,\ \frac{dy}{dx} = 2,\ \frac{d^2y}{dx^2} = e^0\left[2(2) + 1 + 1\right] = 6$$

$$\frac{d^3y}{dx^3} = (12 + 8 + 8 + 1 + 1) = 30$$

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + ---$$

$$= 1 + 2x + 3x^2 + 5x^3 + ---$$

Example 22

Find y in terms of x given that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$ and that $\frac{dy}{dx} = 1$ and $y = 0$ at $x = 0$.

Solution

$$r^2 - 4r + 4 = 0 \Rightarrow r_1 = r_2 = 2$$

$$y_p = kx^2e^{2x}$$

$$y_p' = 2kxe^{2x} + 2kx^2e^{2x}$$

$$y_p'' = 2ke^{2x} + 4kxe^{2x} + 4kxe^{2x} + 4kx^2e^{2x}$$

$$2ke^{2x}+4kxe^{2x}+4kxe^{2x}+4kx^2e^{2x}-8kxe^{2x}-8kx^2e^{2x}+4kx^2e^{2x}\equiv e^{2x}$$
$$\Rightarrow k=\frac{1}{2}$$

$$y=c_1e^{2x}+c_2xe^{2x}+\frac{1}{2}x^2e^{2x}$$

$$0=c_1$$

$$y'=2c_1e^{2x}+c_2e^{2x}+2c_2xe^{2x}+xe^{2x}+x^2e^{2x}\Rightarrow c_2=1$$

$$y=xe^{2x}+\frac{1}{2}x^2e^{2x}$$

Example 23

a) Show that $\frac{1}{2}x\sin x$ is a particular integral of the differential equation

$\frac{d^2y}{dx^2}+y=\cos x$.

b) Hence find the general solution.

Solution

$$y=\frac{1}{2}x\sin x$$
$$y'=\frac{1}{2}\sin x+\frac{1}{2}x\cos x$$

$$y''=\frac{1}{2}\cos x+\frac{1}{2}\cos x-\frac{1}{2}x\sin x$$

$$\frac{d^2y}{dx^2}+y=\cos x-\frac{1}{2}x\sin x+\frac{1}{2}x\sin x=\cos x$$
$\Rightarrow \frac{1}{2}x\sin x$ **is a particular integral of the differential equation** $\frac{d^2y}{dx^2}+y=\cos x$

$$r^2+1=0\Rightarrow r=\pm i$$

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} x \sin x$$

Example 24

a) Given that $x = e^t$, show that $\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

b) Hence show that the substitution $x = e^t$ transforms the differential equation

$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 3$ (I) into $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 3$ (II)

c) Hence solve II and find y in terms of x

Solution

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{1}{x} \Rightarrow \frac{dy}{dt} = x\frac{dy}{dx}$$

$$\frac{dy}{dt} = x\frac{dy}{dx} \Rightarrow \frac{d^2y}{dt^2} = \frac{dx}{dt}\cdot\frac{dy}{dx} + x\frac{d^2y}{dx^2}\cdot\frac{dx}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} = \frac{dy}{dt} + e^{2t}\frac{d^2y}{dx^2}$$

$$\Rightarrow \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) = e^{2t}\frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 3 \Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} - 4\frac{dy}{dt} + 6y = 3$$

$$\Rightarrow \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 3$$

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 3$$

$$r^2 - 5r + 6 = 0$$

$$r = 2, \quad r = 3$$

$$y_p = k \Rightarrow y_p' = y_p'' = 0$$

$$6k = 3 \Rightarrow k = \frac{1}{2}$$

$$y = c_1 e^{2t} + c_2 e^{3t} + \frac{1}{2}$$

$$y = c_1 x^2 + c_2 x^3 + \frac{1}{2}$$

Example 25

Assuming that $\dfrac{d^2y}{dx^2} = 4t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}$

Show that the substitution $x = t^{1/2}$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \left(6x - \frac{1}{x}\right)\frac{dy}{dx} - 16x^2 y = 4x^2 e^{2x^2} \qquad \text{(I)}$$

into the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t} \qquad \text{(II)}$$

Hence find the general solution of (I) giving *y* in terms of *x*.

Solution

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot 2\sqrt{t}$$

Substitute

$$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \left(6\sqrt{t} - \frac{1}{\sqrt{t}}\right)\left(\frac{dy}{dt}\cdot\left(2\sqrt{t}\right)\right) - 16ty = 16te^{2t}$$

$$\Rightarrow 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 12\frac{dy}{dt} - 2\frac{dy}{dt} - 16ty = 16te^{2t}$$

$$\Rightarrow \frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$$

$$r^2 + 3r - 4 = 0 \Rightarrow r = -4, 1 \Rightarrow y_h = Ae^{t} + Be^{-4t}$$

$$y_p = Ce^{2t}$$

$$\Rightarrow 4c+6c-4c=1 \Rightarrow c=\frac{1}{6}$$

$$\Rightarrow y=\mathrm{A}e^{t}+\mathrm{B}e^{-4t}+\frac{1}{6}e^{2t}$$

$$\Rightarrow y=\mathrm{A}e^{x^2}+\mathrm{B}e^{-4x^2}+\frac{1}{6}e^{2x^2}$$

Example 26

a) Find in the form $y=f(x)$, the general solution of the equation: $(x^2-1)\frac{dy}{dx}+xy=1, x>1$

b) i) Given that $y=\frac{u}{x}$, show that $\frac{d^2y}{dx^2}=\frac{1}{x}\frac{d^2y}{dx^2}-\frac{2}{x^2}\frac{du}{dx}+\frac{2u}{x^3}$

ii)Hence find the general solution of the differential equation

$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+25y=0, x>0$

Solution

$$\rho=e^{\int \frac{x}{x^2-1}dx}=e^{\frac{1}{2}\ln(x^2-1)}=\sqrt{x^2-1}$$

$$\sqrt{x^2-1}\,y=\int \frac{1}{\sqrt{x^2-1}}dx=\cosh^{-1}x+C$$

$$y=\frac{\cosh^{-1}x}{\sqrt{x^2-1}}+\frac{C}{\sqrt{x^2-1}}, C\in R$$

$$\frac{dy}{dx}=\frac{\frac{du}{dx}x-u}{x^2}=\frac{1}{x}\frac{du}{dx}-\frac{u}{x^2}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d^2u}{dx^2}x-\frac{du}{dx}}{x^2}-\frac{x^2\frac{du}{dx}-2xu}{x^4}$$

$$= \frac{1}{x}\frac{d^2u}{dx^2} - \frac{1}{x}\frac{d^2u}{dx^2} - \frac{1}{x^2}\frac{du}{dx} + \frac{2u}{x^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2u}{dx^2} - \frac{2}{x^2}\frac{d^2u}{dx^2} + \frac{2u}{x^3}$$

$$\frac{1}{x}\frac{d^2u}{dx^2} - \frac{2}{x^2}\frac{d^2u}{dx^2} + \frac{2u}{x^3} + \frac{2}{x}\left(\frac{1}{x}\frac{du}{dx} - \frac{u}{x^2}\right) + 25\left(\frac{u}{x}\right) = 0$$

$$\Rightarrow \frac{d^2u}{dx^2} + 25u = 0 \Rightarrow u = A\cos 5x + B\sin 5x \Rightarrow y = \frac{1}{x}\left(A\cos 5x + B\sin 5x\right)$$

Example 27

a) Solve the differential equation: $yy' - y' = e^x$

b) Use the substitution $z = \frac{1}{y}$ to solve the differential equation $y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 - y^2 = 0$

Solution

$$dy(y-1) = e^x dx$$

$$\Rightarrow \frac{y^2}{2} - y = e^x + C$$

$$\Rightarrow y^2 - 2y = 2e^x + k, \quad k \in R$$

$$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$\frac{d^2z}{dx^2} = \frac{2}{y^3}\left(\frac{dy}{dx}\right)^2 - \frac{1}{y^2}\frac{d^2y}{dx^2}$$

$$\text{But } \frac{d^2y}{dx^2}\cdot\frac{1}{y^2} - \frac{2}{y^3}\left(\frac{dy}{dx}\right)^2 - \frac{1}{y} = 0 \qquad \text{(from the given)}$$

$$\Rightarrow \frac{d^2z}{dx^2} + z = 0$$

$$\Rightarrow z = (C_1\cos x + C_2\sin x)$$

$$\Rightarrow y = \frac{1}{C_1\cos x + C_2\sin x}$$

Example 28

Find the general solution of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=6$

Solution

$$\mathbf{r^2+2r-3=0}$$

$$\Rightarrow \mathbf{r=-3,1}$$

$$\Rightarrow y_h = C_1e^{x}+C_2e^{-3x}$$

$$y_p=-2$$

$$\Rightarrow y=C_1e^{x}+C_2e^{-3x}-2$$

Example 29

a) Find in the form $y = f(x)$, the general solution of the equation: $\left(x^2-1\right)\frac{dy}{dx}+xy=1, \quad x>1$

b) i) Given that $y=\frac{u}{x}$, show that

$$\frac{d^2y}{dx^2}=\frac{1}{x}\frac{d^2u}{dx^2}-\frac{2}{x^2}\frac{du}{dx}+\frac{2u}{x^3}$$

ii) Hence find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+25y=0,\ x>0$$

Solution

$$\rho=e^{\int\frac{x}{x^2-1}dx}=e^{\frac{1}{2}\ln\left(x^2-1\right)}=\sqrt{x^2-1}$$

$$\sqrt{x^2-1}\,y=\int\frac{1}{\sqrt{x^2-1}}dx=\cosh^{-1}x+C$$

Example 30

a) Show that the transformation $y = xv$ transforms the equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(2 + 9x^2\right)y = x^5 \qquad \text{(I)}$$

into the equation

$$\frac{d^2v}{dx^2} + 9v = x^2 \qquad \text{(II)}$$

b) Solve the differential equation II to find v as a function of x.

c) Hence, state the general solution of the differential equation I.

Solution

$$y = xv$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

and $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx} + \dfrac{dv}{dx} + x\dfrac{d^2v}{dx^2} = 2\dfrac{dv}{dx} + x\dfrac{d^2v}{dx^2}$

Substitute: $x^2\left(2\dfrac{dv}{dx} + x\dfrac{d^2v}{dx^2}\right) - 2x\left(v + x\dfrac{dv}{dx}\right) + \left(2 + 9x^2\right)y = x^5$

$$\Rightarrow \frac{d^2v}{dx^2} + 9v = x^2$$

$$r^2 + 9 = 0 \Rightarrow r = \mp 3i$$

$$\Rightarrow v_h = A\cos 3x + B\sin 3x$$

$$v_p = Cx^2 + Dx + E$$

$$\Rightarrow 2C + 9Cx^2 + 9Dx + 9E = x^2 \Rightarrow C = \frac{1}{9}, D = 0, E = -\frac{2}{81}$$

$$\Rightarrow v = A\cos 3x + B\sin 3x + \frac{1}{9}x^2 - \frac{2}{81}$$

$$\Rightarrow y = \mathbf{A}x\cos 3x + \mathbf{B}x\sin 3x + \frac{1}{9}x^3 - \frac{2}{81}x, \ \mathbf{A} \text{ and } \mathbf{B} \in \mathbb{R}$$

Example 31

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 36x$.

Solution

$$r^2 + r = 0$$
$$\Rightarrow r = 0, -1$$

$$\Rightarrow y_h = Ae^{0x} + Be^{-x} = A + Be^{-x}$$

$$y_p = Cx^2 + Dx$$

$$y_p' = 2Cx + D \text{ and } y_p'' = 2C$$
$$\Rightarrow 2C + 2Cx + D \equiv 36x$$

$$\Rightarrow 2C + D = 0 \text{ and } 2C = 36 \Rightarrow C = 18 \text{ and } D = -36$$

$$\Rightarrow y = A + Be^{-x} + 18x^2 - 36x$$

Example 32

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$$

Given that $x = 0$ and $\frac{dx}{dt} = 2$ at $t = 0$,

a) find x in terms of t.

The solution to part (a) is used to represent the motion of a particle P on the x-axis. At time t seconds, where $t > 0$, P is x meters from the origin O.

b) Show that the maximum distance between O and P is $\frac{2\sqrt{3}}{9}$ and justify that this distance is a maximum.

<u>Solution</u>

$$r^2+5r+6=0$$

$$(r+3)(r+2)=0$$

$$\Rightarrow r=-3,-2$$

$$\Rightarrow x_h = Ae^{-3t}+Be^{-2t}$$

$$x_p = Ce^{-t} \Rightarrow Ce^{-t}-5Ce^{-t}+6Ce^{-t}=2e^{-t}$$

$$\Rightarrow C=1$$

$$\Rightarrow x = Ae^{-3t}+Be^{-2t}+e^{-t}$$

$$\left.\begin{array}{l} 0=A+B+1 \\ 2=-3A-2B-2B-1 \end{array}\right\}$$

$$\Rightarrow A=-1,\ B=0$$

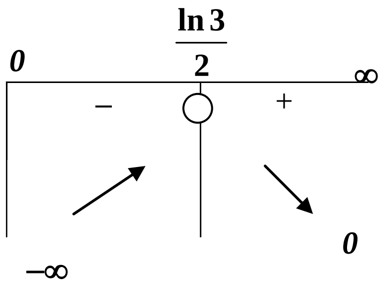
$$\Rightarrow \boxed{x=-e^{-3t}+e^{-t}}$$

$$\frac{dx}{dt}=3e^{-3t}-e^{-t}$$

$$\frac{dx}{dt}=0 \Rightarrow 3e^{-3t}-e^{-t}=0$$

$$\Rightarrow 3e^{t}=e^{3t} \Rightarrow e^{2t}=3$$

$$\Rightarrow t=\frac{\ln 3}{2}$$



$$\mathbf{P_{max} = -e^{-\frac{3}{2}\ln 3} + e^{-\frac{\ln 3}{2}}}$$
$$= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$$

Example 33

a) Solve the differential equation: $yy' - y' = e^x$

b) Use the substitution $z = \frac{1}{y}$ to solve the differential equation

$$y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 - y^2 = 0.$$

<u>**Solution**</u>

$$y'(y-1) = e^x$$

$$(y-1)\,dy = e^x dx$$
$$\Rightarrow \frac{y^2}{2} - y = e^x + C,\ C \in \mathbb{R}$$

$$\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$\frac{d^2z}{dx^2} = \frac{2}{y^3}\left(\frac{dy}{dx}\right)^2 - \frac{1}{y^2}\frac{d^2y}{dx^2}$$

Substitute in the given you get
$$\frac{d^2z}{dx^2} + z = 0$$

$$\Rightarrow z = C_1\cos x + C_2\sin x$$

$$\Rightarrow y = \frac{1}{C_1\cos x + C_2\sin x},\quad C_1, C_2 \in \mathbb{R}$$

Example 34

Given that $y=2$ at $x=0$ and $\frac{dy}{dx}=-5$ at $x=0$, find *y* in terms of *x*, if given further that $\frac{d^2y}{dx^2}+\frac{dy}{dx}=2x+3$.

Solution

$$\boldsymbol{r^2+r=0}$$
$$\boldsymbol{r=0,-1}$$

$$\boldsymbol{y_h = C_1e^{0x}+C_2e^{-1x}}$$

$$\boldsymbol{y_p = Ax^2+Bx+C}$$

$$\boldsymbol{y_p' = 2Ax,\ \ y_p'' = 2A}$$
$$\boldsymbol{\Rightarrow 2A+2Ax+B=2x+3}$$
$$\boldsymbol{\Rightarrow A=1, B=1}$$

$$\boldsymbol{\Rightarrow y = C_1+C_2e^{-x}+x^2+x}$$

$$\boldsymbol{\left.\begin{array}{l}C_1+C_2=2\\ -C_2+1=-5\end{array}\right\} \Rightarrow \left.\begin{array}{l}C_2=6\\ C_1=-4\end{array}\right\}}$$

$$\boldsymbol{\Rightarrow y = 6e^{-x}+x^2+x-4}$$

Example 35

Transform the equation $\frac{d^2y}{dx^2}+x^2+y+2=0$ to $\frac{d^2t}{dx^2}+1=0$, by means of the substitution $y=t-x^2$, and hence find the general solution.

Solution

$$\boldsymbol{\frac{dy}{dx}=\frac{dt}{dx}-2x}$$

$$\boldsymbol{\frac{d^2y}{dx^2}=\frac{d^2t}{dx^2}-2}$$

$$\Rightarrow \frac{d^2t}{dx^2} - 2 + x^2 + t - x^2 + 2 = 0$$

$$\Rightarrow \frac{d^2t}{dx^2} + t = 0$$

$$\Rightarrow t = A\cos x + B\sin x$$

$$\Rightarrow y = A\cos x + B\sin x - x^2$$

Example 36

Given that $y = \frac{u}{x}$, show that $\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2u}{dx^2} - \frac{2}{x^2}\frac{du}{dx} + \frac{2u}{x^3}$. Hence find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + 25y = 0, \quad x > 0$.

Solution

$$\frac{dy}{dx} = \frac{x\frac{du}{dx} - u}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\left(x\frac{d^2u}{dx^2} + \frac{du}{dx} - \frac{du}{dx}\right)x^2 - 2x\left(x\frac{du}{dx} - u\right)}{x^4}$$

$$= \frac{x^3\frac{d^2u}{dx^2} - 2x^2\frac{du}{dx} + 2xu}{x^4}$$

$$= \frac{1}{x}\frac{d^2u}{dx^2} - \frac{2}{x^2}\frac{du}{dx} + \frac{2u}{x^3}$$

Substitute in the given equation we get

$$\frac{d^2u}{dx^2} + 25u = 0$$

$$\Rightarrow u = A\cos 5x + B\sin 5x$$

$$\Rightarrow y = \frac{1}{x}\left(A\cos x + B\sin x\right), \quad A, B \in \mathbb{R}$$

Example 37

$\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + y = 0$, with $y = 2$ at $x = 0$ and $\frac{dy}{dx} = 1$ at $s = 0$

a) Use the Taylor series method to express y as a polynomial in ascending powers of x up to and including the term in x^3.

b) Show that at $x = 0$, $\frac{d^4y}{dx^4} = 0$

<u>Solution</u>

$$\frac{d^3y}{dx^3} + 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$\Rightarrow \text{at } x = 0,\ y = 2,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = -2,$$

$$\frac{d^3y}{dx^3} = -1$$

$$\Rightarrow y = 2 + x - \frac{2x^2}{2!} - \frac{x^3}{3!}$$

$$\Rightarrow y = 2 + x - x^2 - \frac{x^3}{6}$$

$$\frac{d^4y}{dx^4} + 2\frac{dy}{dx} + 2x\frac{d^2y}{dx^2} + 2x\frac{d^2y}{dx^2} + x^2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} + 2 + 0 + 0 + 0 - 2 = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} = 0$$

Example 38

Solve the differential equation $4\frac{d^2y}{dx^2} + y = 0$ given that $y = 2$ and $\frac{dy}{dx} = -1$ at $x = 0$.

<u>Solution</u>

$$4r^2 + 1 = 0$$

$$\Rightarrow r = \pm \frac{1}{2}\mathrm{i}$$

$$y = C_1 \cos \frac{1}{2} x + C_2 \sin \frac{1}{2} x$$

when $x = 0$ **and** $y = 2 \Rightarrow 2 = C_1$

$$\frac{dy}{dx} = -\frac{1}{2} C_1 \sin \frac{1}{2} x + \frac{1}{2} C_2 \cos \frac{1}{2} x$$
$$\Rightarrow -1 = 0 + \frac{1}{2} C_2 \Rightarrow C_2 = -2$$

$$y = 2\cos \frac{1}{2} x - 2\sin \frac{1}{2} x$$

Example 39

Determine the general solution of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 8\sin x$

Solution

$$r^2 - 4r + 5 = 0 \Rightarrow r = 2 \pm \mathrm{i}$$

$$y = e^{2x}\left(C_1 \cos x + C_2 \sin x\right)$$

$$y_P = A\cos x + B\sin x$$

$$y_P' = -A\sin x + B\cos x$$

$$y_P'' = -A\cos x - B\cos x$$

$$-A\cos x - B\sin x + 4A\sin x - 4B\cos x + 5A\cos x + 5B\sin x \equiv 8\sin x$$
$$\left(4A - 4B\right)\cos x + \left(4B + 4A\right)\sin x \equiv 8\sin x$$

$$\left.\begin{array}{l} 4A - 4B = 0 \\ 4A + 4B = 8 \end{array}\right\} \Rightarrow A = B = 1$$

$$y = e^{2x}\left(C_1 \cos x + C_2 \sin x\right) + \cos x + \sin x$$

Example 40

Find the general solution of the differential equation $\frac{d^2y}{dx^2}+5\frac{dy}{dx}-6y=36x$

Solution

$\boldsymbol{r^2+5r-6=0}$

$\boldsymbol{\Rightarrow r=1,\ r=-6}$

$\boldsymbol{y=C_1e^{x}+C_2e^{-6x}+y_P}$

$\boldsymbol{y_P=ax+b,}$

$\boldsymbol{y'=a,}$

$\boldsymbol{y''_P=0}$

$\boldsymbol{5a-6ax-6b=36x\Rightarrow -6ax+5a-6b=36x}$

$\boldsymbol{-6a=36\Rightarrow a=-6}$

$\boldsymbol{b=\frac{5a}{6}=-5}$

$\boldsymbol{y=C_1e^{x}+C_2e^{-6x}-6x-5}$

Example 41

Given that $x=\ln t$, $t>0$ and that y is a function of x

a) Find $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and t

b) Show that $\frac{d^2y}{dx^2}=t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}$

c) Show that the substitution $x=\ln t$ transforms the differential equation

$\frac{d^2y}{dx^2}-\left(1-6e^{x}\right)\frac{dy}{dx}+10ye^{2x}=5e^{2x}\sin 2e^{x}$ (I) into the differential equation

$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+10y=5\sin 2t$ (II)

d) Hence find the general solution of (I) giving your answer in the form $y = f\left(x\right)$

Solution

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{1/t} = t\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dx}\right) = \frac{-1}{t^2} \cdot \frac{dy}{dx} + \frac{1}{t}\frac{d^2y}{dx^2} \cdot \frac{1}{t}$$
$$= \frac{-1}{t^2} \cdot \frac{dy}{dx} + \frac{1}{t^2} \cdot \frac{d^2y}{dx^2}$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{1}{t^2}\frac{dy}{dx} = \frac{1}{t^2}\frac{d^2y}{dx^2}$$
$$\Rightarrow \frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + \frac{dy}{dx} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$$

$$t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2} - \left(1 - 6e^{\ln t}\right)\left(t\frac{dy}{dt}\right) + 10ye^{2\ln t} = 5e^{2\ln t}\sin 2e^{\ln t}$$
$$\Rightarrow t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} + 6t^2\frac{dy}{dt} + 10t^2y = 5t^2\sin 2t$$
$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 5\sin 2t$$

$$r^2 + 6r + 10 = 0$$
$$r = -3 \pm i$$

$$\Rightarrow y_h = e^{-3t}\left(C_1\cos t + C_2\sin t\right)$$

$$y_p = A\sin 2t + B\cos 2t,$$

find y_p' and y_p''

and susbtituting in the equation we get

$$A = \frac{1}{6},$$

$$B = -\frac{1}{3}$$

$$\Rightarrow y = e^{-3t}\left(C_1\cos t + C_2\sin t\right) + \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t$$
$$\Rightarrow y = e^{-3e^x}\left(C_1\cos e^x + C_2\sin e^x\right) + \frac{1}{6}\sin 2e^x - \frac{1}{3}\cos 2e^x$$

Example 42

Given that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=2x+3$ and y = 2 at x = 0 and $\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ at x = 0, find y in terms of x.

<u>Solution</u>

$\boldsymbol{r^2 + r = 0 \Rightarrow r = 0, -1}$

$\boldsymbol{\Rightarrow y_n = C_1 e^{0x} + C_2 e^{-x} = C_1 + C_2 e^{-x}}$

$\boldsymbol{y_p = ax^2 + bx}$

$\boldsymbol{\Rightarrow 2a + 2ax + b \equiv 2x + 3}$

$\boldsymbol{\Rightarrow a = 1}$

and b = 1

$\boldsymbol{\Rightarrow y = x^2 + x + C_1 + C_2 e^{-x}}$

$\boldsymbol{y = 2 \text{ at } x = 0}$

$\boldsymbol{\Rightarrow C_1 + C_2 = 2}$

$\boldsymbol{\frac{dy}{dx} = -5 \text{ at } x=0}$

$\boldsymbol{\Rightarrow -5 = 1 - C_2}$

$\boldsymbol{\Rightarrow C_2 = 6}$

$\boldsymbol{C_1 = -4}$

$\boldsymbol{\Rightarrow y = x^2 + x + 6e^{-x} - 4}$

Example 43

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+4\frac{\mathrm{d}y}{\mathrm{d}x}+5y=65\sin 2x, \quad x>0$$

Find the general solution of the differential equation.

<u>Solution</u>

$$\boldsymbol{r^2+4r+5=0}$$

$$\Rightarrow \boldsymbol{r=-2\mp\sqrt{-1}=-2\mp i}$$

$$\Rightarrow y_h = e^{-2x}\left(M\cos x + N\sin x\right)$$

$$y_p = A\sin 2x + B\cos 2x$$

$$y_p' = 2A\cos 2x - 2B\sin 2x$$

$$y_p'' = -4A\sin 2x - 4B\cos 2x$$

$$\left.\begin{array}{l} -4A-8B+5A=65 \\ -4B+8A+5B=0 \end{array}\right\} \Rightarrow A=1,$$

$$B=-8$$

$$\Rightarrow y = \sin 2x - 8\cos 2x + e^{-2x}\left(M\cos x + N\sin x\right)$$

Example 44

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t}, \quad t \ge 0$$

a) Show that Kt^2e^{3t} is a particular integral of the differential equation, where K is a constant to be found.

b) Find the general solution of the differential equation.

Given that a particular solution satisfies y = 3 and $\frac{dy}{dt}=1$ when t = 0,

c) Find this solution.

Solution

$$y' = 2K\,te^{3t} + 3Kt^2e^{3t}$$

$$y'' = 2Ke^{3t} + 6Kte^{3t} + 6Kte^{3t} + 9Kt^2e^{3t}$$

Substitute you get $K=2$

$$r^2 - 6r + 9 = 0 \Rightarrow r = 3$$

$$\Rightarrow y_h = e^{3t}(A + Bt)$$

$$\Rightarrow y = (A + Bt)e^{3t} + 2t^2e^{3t}$$

$$A = 3$$

$$B = -8 \Rightarrow y = \left(3 - 8t + 2t^2\right)e^{3t}$$

Example 45

i) Given that $y = \frac{u}{x}$, show that

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2u}{dx^2} - \frac{2}{x^2}\frac{du}{dx} + \frac{2u}{x^3}$$

ii) Hence find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + 25y = 0, \ \ x > 0\,.$$

<u>Solution</u>

$$y = \frac{u}{x}$$

$$\frac{dy}{dx} = \frac{x\frac{du}{dx} - u}{x^2} = \frac{1}{x}\frac{du}{dx} - \frac{u}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{du}{dx} + \frac{1}{x}\frac{d^2u}{dx^2} -$$

$$-\frac{x^2\frac{du}{dx} - 2xu}{x^4}$$

$$= -\frac{1}{x^2}\frac{du}{dx} + \frac{1}{x}\frac{d^2u}{dx^2} - \frac{1}{x^2}\frac{du}{dx} + \frac{2}{x^3}u$$

$$= \frac{1}{x}\frac{d^2u}{dx^2} - \frac{2}{x^2}\frac{du}{dx} + \frac{2u}{x^3}$$

$$\frac{1}{x}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\frac{2}{x^2}\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{2u}{x^3}+\frac{2}{x^2}\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{2u}{x^3}+\frac{25u}{x}=0$$

$$\Rightarrow \frac{\mathrm{d}^2u}{\mathrm{d}x^2}+25u=0$$

$$\Rightarrow u=\mathrm{C}_1\cos 5x+\mathrm{C}_2\sin 5x$$

$$\Rightarrow yx=\mathrm{C}_1\cos 5x+\mathrm{C}_2\sin 5x$$

Example 46

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^x\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2+1\right)$$

a) Show that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\mathrm{e}^x\left(2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\mathrm{k}y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2+1\right)$

where k is a constant to be found.

Given that $x=0$, $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=2$

b) Find a series solution for y in ascending powers of x, up to and including the term in x^3.

Solution

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^x\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2+1\right)$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\mathrm{e}^x\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2+1\right)+\mathrm{e}^x\left[2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right]$$

$$=\mathrm{e}^x\left(2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+4y\frac{\mathrm{d}y}{\mathrm{d}x}+y^2+1\right)$$

$$\Rightarrow \mathrm{k}=4$$

$$x=0,\ y=1,\ \frac{\mathrm{d}y}{\mathrm{d}x}=2,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^0\left[2(2)+1+1\right]=6$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=(12+8+8+1+1)=30$$

$$y=y(0)+\frac{y'(0)}{1!}x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\ldots$$

$$=1+2x+3x^2+5x^3+\ldots$$

Differential Equations Practice Questions

Test

1. The general solution of $y'' - 4y' - 5y = 0$ is

[-A-] $y = Ae^{x} + Be^{5x}$

[-B-] $y = Ae^{x} + Be^{-5x}$

[-C-] $y = e^{2x}\left[A\cos x + B\sin x\right]$

[-D-] $y = Ae^{-x} + Be^{5x}$

[-E-] $y = e^{2x}\left[A\cos 2x + B\sin 2x\right]$

2. The general solution of $y'' + 4y' = 0$ is

[-A-] $y = A(\cos 2x + B)$

[-B-] $y = Ae^{2x} + Bxe^{2x}$

[-C-] $y = A\cos 2x + B\sin 2x$

[-D-] $y = Ae^{-4x} + B$

[-E-] $y = Ae^{2x} + Be^{-2x}$

3. A particular solution of the non-homogenous linear differential equation $y'' + 3y' + 2y = 2x^2 + 4x$ is

[-A-] $y = x^2 - x$

[-B-] $y = 2x^2 + x + \frac{1}{2}$

[-C-] $y = x^2 - x + \frac{1}{2}$

[-D-] $y = e^{-x} + e^{-2x}$

[-E-] $y = e^{-x} + x^2 - \frac{1}{2}$

4. A particular solution of $y'' - 4y = 3\cos x$ is $\frac{-3}{5}\cos x$. The general solution of the equation is

[-A-] $\mathbf{y = Ae^{2x} + Bxe^{2x} - \frac{3}{5}\cos x}$

[-B-] $\mathbf{y = A + Be^{4x} - \frac{3}{5}\cos x}$

[-C-] $\mathbf{y = Ae^{2x} + Be^{-2x} - \frac{3}{5}\cos x}$

[-D-] $\mathbf{y = A\cos 2x + B\sin 2x - \frac{3}{5}\cos x}$

[-E-] None of the above

5. The general solution of $y'' - 2y' = 4$ is

[-A-] $\mathbf{y = -2x + C}$

[-B-] $\mathbf{y = -2x + Ae^{2x} + B}$

[-C-] $\mathbf{y = Ae^{2x} + B}$

[-D-] $\mathbf{y = 2x^2 + Ae^{2x} + B}$

[-E-] None of the above

6. If $\frac{dx}{dt} = kx$ and if $x = 2$ when $t = 0$ and $x = 6$ when $t = 1$, then $k =$

[-A-] $\mathbf{\ln 4}$

[-B-] 8

[-C-] $\mathbf{e^3}$

[-D-] 3

[-E-] none of the above

7. If $y = f(x)$ is a solution of the differential equation $y' - y = 2e^x$ and if $f(0) = -3$, then $f(1) =$

[-A-] $-\mathbf{e}$

[-B-] $-\mathbf{2e}$

[-C-] $\dfrac{-\mathbf{3}}{\mathbf{e}}$

[-D-] $\dfrac{\mathbf{3+e}}{\mathbf{e}}$

[-E-] $-\mathbf{1}$

8. The general solution of $y'' + y' = 4$ is

[-A-] $y = 4x + C$

[-B-] $y = 4x + Ae^{-x} + B$

[-C-] $y = 4x + Ae^{x} + B$

[-D-] $y = 4x + A\cos x + B\sin x$

[-E-] $y = 4x + Ae^{x} + Be^{-x}$

Show your work.

1. Solve $y' = 2x(1 - y)$ using an integrating factor and then solve again using the method of separation of variables. Show that the answer in the latter method is reducible to the first answer.

2. Solve $\frac{dy}{dx} + \frac{2}{x+1}y = \frac{x}{x+1}, x > -1, y(0) = 1.$

3. A cup of coffee when first poured had a temperature of 198°F. Four minutes later, its temperature dropped to 140°F. If the temperature of the room where the coffee was served is 82°F, what would the temperature of the coffee be 12 minutes after it had been served?

4. Solve the initial-value problem $y'' + 4y' + 5y = 0, y(0) = 1, y'(0) = -2$.

5. Solve the differential equation $y'' + 4y' + 5y = e^{-2x}$.

6. Consider the differential equation $xy'' + y' = x, y(1) = 1, y'(1) = -2$. Use the substitution $u = y'$ to reduce the equation to a first order differential equation, solve the resulting equation, and deduce the solution of the original equation.

7. Given that the differential equation $y'' + 4xy' + p(x)y = 0$ has two solutions $y_1 = u(x)$ and $y_2 = xu(x)$, $u(0) = 1$. Find $p(x)$ and $u(x)$.

8. a) If y is a function of x and $x = e^t$, show that $\frac{dy}{dt} = x\frac{dy}{dx}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

b) Use the change of variable in part a) to solve $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$.

9. Show that the substitution $y' = u(y)$ reduces the differential equation $y\frac{d^2y}{dx^2} - y^2\left(\frac{dy}{dx}\right) - \left(\frac{dy}{dx}\right)^2 = 0$ to a first order linear differential equation. Solve the resulting equation and deduce the solution of the original equation.

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