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BEST BILINEAR APPROXIMATIONS OF THE CLASSES $S^{\Omega}_{p,\theta}B$ OF PERIODIC FUNCTIONS OF MANY VARIABLES

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We obtain exact-order estimates for the best bilinear approximations of the classes $S^{\Omega}_{p,\theta}B$ of periodic functions of many variables in the space L_q under certain restrictions on the parameters p, q, and θ.

Introduction

This paper is devoted to the investigation of the best bilinear approximations of periodic functions of many variables in the space L_q under certain restrictions on the parameters p, q, and θ. The paper consists of the introduction and two sections. In the introduction, we give necessary notation and the definitions of classes under investigation. Section 1 is auxiliary. In particular, we formulate and prove there a theorem on estimates for the best M-term trigonometric approximations. The obtained results are used in Sec. 2 for finding upper bounds for the best bilinear approximations of functions of $2d$ variables of the form $f(x-y)$, $x, y \in \pi_d$, generated by functions $f(x) \in S^{\Omega}_{p,\theta}B$.

We now give necessary notation and definitions.

Let $\mathbb{R}^d$, $d \geq 1$, be the d-dimensional Euclidean space with elements $x = (x_1, \ldots, x_d)$ and let $L_p(\pi_d)$, $\pi_d = \prod_{j=1}^{d} [-\pi; \pi]$, be the space of functions $f(x) = f(x_1, \ldots, x_d)$ 2π-periodic in each variable and summable to the power p, $1 \leq p < \infty$ (essentially bounded for $p = \infty$). The norm in this space is defined as follows:

$$\|f\|_p = \left((2\pi)^{-d} \int\limits_{\pi_d} |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \pi_d} |f(x)|.$$

Denote a subset of functions $f \in L_p(\pi_d)$ that satisfy the condition

$$\int\limits_{-\pi}^{\pi} f(x) dx_j = 0, \quad j = \overline{1, d},$$

by $L^{\circ}_p(\pi_d)$.

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We now define spaces $S^{\Omega}_{p,\theta}B \subset L_p(\pi_d)$ whose properties are determined by a majorant function $\Omega(t)$, $t = (t_1, \ldots, t_d) \in \mathbb{R}^d_+$, for the mixed modulus of continuity of order l $(l \in \mathbb{N})$ of a function $f \in L_p(\pi_d)$ and numerical parameters p and θ, $1 \le p, \theta \le \infty$.

Thus, for an arbitrary function $f \in L_p(\pi_d)$, we consider its mixed modulus of continuity of order l, namely

$$\Omega_l(f,t)_p = \sup_{\substack{|h_j| \le t_j \\ j=\overline{1,d}}} \|\Delta^l_h f(\cdot)\|_p,$$

where

$$\Delta^l_h f(x) = \Delta^l_{h_d} \ldots \Delta^l_{h_1} f(x) = \Delta^l_{h_d}(\ldots(\Delta^l_{h_1} f(x))), \quad h = (h_1, \ldots, h_d),$$

is the mixed lth difference with step h_j with respect to the variable x_j, $j = \overline{1,d}$, and

$$\Delta^l_{h_j} f(x) = \sum_{n=o}^{l} (-1)^{l-n} C^n_l f(x_1, \ldots, x_{j-1}, x_j + nh_j, x_{j+1}, \ldots x_d).$$

Let $\Omega(t) = \Omega(t_1, \ldots, t_d)$ be a given function of the type of a mixed modulus of continuity of order l that satisfies the following conditions:

1. $\Omega(t) > 0$, $t_j > 0$, $j = \overline{1,d}$, $\Omega(t) = 0$, and $\prod_{j=1}^{d} t_j = 0$.

2. $\Omega(t)$ is continuous on $\mathbb{R}^d_+$.

3. $\Omega(t)$ does not decrease in each variable $t_j \ge 0$, $j = \overline{1,d}$, for any fixed values of the other variables t_i, $i \ne j$.

4. $\Omega(m_1 t_1, \ldots, m_d t_d) \le C \left(\prod_{j=1}^{d} m_j \right)^l \Omega(t)$, where $m_j \in \mathbb{N}$, $j = \overline{1,d}$, and $C > 0$ is a certain constant.

Denote the set of these functions Ω by $\Psi_{l,d}$. For $d = 1$, we write Ψ_l. Note that if $f \in L_p(\pi_d)$, then $\Omega_l(f, \cdot) \in \Psi_{l,d}$.

We impose additional conditions on the functions $\Omega \in \Psi_{l,d}$. We describe these conditions by using the following two concepts introduced by Bernshtein in [1]:

(a) a nonnegative function $\varphi(\tau)$, $\tau \in [0;\infty)$, almost increases if there exists a constant $C_1 > 0$ such that $\varphi(\tau_1) \le C_1 \varphi(\tau_2)$ for any τ_1 and τ_2, $0 \le \tau_1 < \tau_2$;

(b) a positive function $\varphi(\tau)$, $\tau \in (0;\infty)$, almost decreases if there exists a constant $C_2 > 0$ such that $\varphi(\tau_1) \ge C_2 \varphi(\tau_2)$ for any τ_1 and τ_2, $0 < \tau_1 < \tau_2$.

Assume that $d = 1$ and $\Omega \in \Psi^{(1,2)}_l$, i.e., for $\Omega(t)$, $t \ge 0$, at least conditions 1 and 2 are satisfied.

We write

(i) $\Omega \in S^{\alpha}$, $\alpha > 0$, if the function $\dfrac{\Omega(\tau)}{\tau^{\alpha}}$ almost increases for $\tau > 0$;

(ii) $\Omega \in S_l$ if there exists γ, $0 < \gamma < l$, such that the function $\dfrac{\Omega(\tau)}{\tau^{\gamma}}$ almost decreases for $\tau > 0$.

The conditions for the function Ω to belong to the sets S^{α} and S_l are often called the Bari–Stechkin conditions [2].

In the case where $d > 1$, we assume for a function $\Omega \in \Psi_{l,d}^{(1,2)}$ that $\Omega \in S^{\alpha}$ (respectively, $\Omega \in S_l$, $l \in \mathbb{N}$), $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_j > 0$, $j = \overline{1,d}$, if $\Omega(t_1, \ldots, t_d)$, regarded as a function of t_j, $j = \overline{1,d}$, belongs to the set S^{α_j} (respectively, S_l) for any values of the other variables t_i, $i \neq j$, .

We also set $\Phi_{\alpha,l}^{d} = \Psi_{l,d} \cap S^{\alpha} \cap S_l$.

Thus, let $1 \le p, \theta \le \infty$ and $\Omega \in \Phi_{\alpha,l}^{d}$. Then

$$S_{p,\theta}^{\Omega} B = \left\{ f \in L_p(\pi_d) \colon |f|_{S_{p,\theta}^{\Omega} B} < \infty \right\},$$

where the seminorm $|f|_{S_{p,\theta}^{\Omega} B}$ is defined by the relation

$$|f|_{S_{p,\theta}^{\Omega} B} = \begin{cases} \left(\int\limits_{\pi_d} \left(\dfrac{\Omega_l(f,t)_p}{\Omega(t)} \right)^{\theta} \prod\limits_{j=1}^{d} \dfrac{dt_j}{t_j} \right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup\limits_{t \ge 0} \dfrac{\Omega_l(f,t)_p}{\Omega(t)}, & \theta = \infty. \end{cases} \tag{1}$$

We define the norm in the space $S_{p,\theta}^{\Omega} B$ as follows:

$$\|f\|_{S_{p,\theta}^{\Omega} B} := \|f\|_p + |f|_{S_{p,\theta}^{\Omega} B}, \quad 1 \le p, \theta \le \infty.$$

The definition of the spaces $S_{p,\theta}^{\Omega} B$ presented above is taken (with slight modification) from [3]. For $\theta = \infty$, the spaces $S_{p,\theta}^{\Omega} B$ (denoted by $S_p^{\Omega} H$) were introduced in [4].

The scale of spaces $S_{p,\theta}^{\Omega} B$ is a natural generalization of the scale of Nikol'skii–Besov spaces $B_{p,\theta}^{r}$, $r = (r_1, \ldots, r_d)$, $r_j > 0$, $j = \overline{1,d}$ (see, e.g., [5]), and $S_{p,\theta}^{\Omega} B \equiv B_{p,\theta}^{r}$ for $\Omega(t) = \prod_{j=1}^{d} t_j^{r_j}$, $r_j < l$, $j = \overline{1,d}$ (note that, for $\theta = \infty$, $B_{p,\theta}^{r}$ are the Nikol'skii spaces H_p^r [6]).

In what follows, we use order relations. The notation $A \asymp B$ means a two-sided inequality between expressions A and B, i.e., $C_3 B \le A \le C_4 B$, where $C_3, C_4 > 0$ are constants whose values may be different in different relations. If $A \le C_5 B$, $C_5 > 0$, and $A \ge C_6 B$, $C_6 > 0$, then we write $A \ll B$ and $A \gg B$, respectively. The dependence of these constants on the corresponding parameters follows from the context. We do not focus our attention on this in using the symbols $\asymp$, $\ll$, and $\gg$.

We now formulate several known statements related to an equivalent representation of the norm $\|f\|_{S_{p,\theta}^{\Omega} B}$ of $f \in S_{p,\theta}^{\Omega} B$, $1 \le p, \theta \le \infty$, $\Omega \in \Phi_{\alpha,l}^{d}$, and necessary for the proof of our results. These representations are given in terms of the defined order of growth of the p-norms of certain trigonometric polynomials constructed on the basis of the expansion of a function $f \in L_p(\pi_d)$ in the Fourier series in a trigonometric system.

Thus, assume that $f \in L_p(\pi_d)$,

$$\delta_s(f,x) = \sum_{k \in \rho(s)} \hat{f}(k) e^{i(k,x)}, \quad (k,x) = k_1 x_1 + \ldots + k_d x_d,$$

where

$$\hat{f}(k) = (2\pi)^{-d} \int\limits_{\pi_d} f(t) e^{-i(k,t)} dt$$

are the Fourier coefficients of the function f, and, for every vector $s = (s_1, \ldots, s_d)$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$,

$$\rho(s) := \left\{ k = (k_1, \ldots, k_d) \in \mathbb{Z}^d : 2^{s_j - 1} \le |k_j| < 2^{s_j},\ j = \overline{1,d} \right\}.$$

It was established in [3] that, for $1 < p < \infty$, $1 \le \theta \le \infty$, $\Omega \in \Phi^d_{\alpha,l}$, and $f \in S^{\Omega}_{p,\theta}B \cap L^{\circ}_p(\pi_d)$, one has

$$\|f\|_{S^{\Omega}_{p,\theta}B} \asymp \begin{cases} \left(\sum_s \Omega(2^{-s})^{-\theta} \|\delta_s(f,\cdot)\|_p^{\theta} \right)^{1/\theta}, & 1 \le \theta < \infty, \\ \sup\limits_s \dfrac{\|\delta_s(f,\cdot)\|_p}{\Omega(2^{-s})}, & \theta = \infty, \end{cases} \tag{2}$$

where $\Omega(2^{-s}) = \Omega(2^{-s_1}, \ldots, 2^{-s_d})$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$.

One can see that this representation of the norm does not include the cases $p = 1$ and $p = \infty$. A certain modification of the right-hand side of (2) enables one to establish an analogous representation that includes these cases.

Let

$$V_n(t) = 1 + 2 \sum_{k=1}^{n} \cos kt + 2 \sum_{k=n+1}^{2n-1} \left(\frac{2n-k}{n} \right) \cos kt$$

be the de la Vallée-Poussin kernel of order $2n$ and let, at a point $x = (x_1, \ldots, x_d)$,

$$A_s(x) = \prod_{j=1}^{d} \left(V_{2^{s_j}}(x_j) - V_{2^{s_j - 1}}(x_j) \right), \quad s = (s_1, \ldots, s_d), \quad s_j \in \mathbb{N}, \quad j = \overline{1,d}. \tag{3}$$

If $f \in L_p(\pi_d)$, $1 \le p \le \infty$, then we set

$$A_s(f,x) := f * A_s.$$

It was established in [7] that, for $1 \le p \le \infty$, $1 \le \theta < \infty$, $\Omega \in \Phi^d_{\alpha,l}$, and $f \in S^{\Omega}_{p,\theta}B \cap L^{\circ}_p(\pi_d)$, one has

$$\|f\|_{S^{\Omega}_{p,\theta}B} \asymp \left(\sum_s \Omega(2^{-s})^{-\theta} \|A_s(f,\cdot)\|_p^{\theta} \right)^{1/\theta}, \quad 1 \le \theta < \infty. \tag{4}$$

For $\theta = \infty$, the following relation is true [4]:

$$\|f\|_{S^{\Omega}_{p,\infty}B} \asymp \sup_s \frac{\|A_s(f,\cdot)\|_p}{\Omega(2^{-s})}. \tag{5}$$

In what follows, we use the spaces $S^{\Omega}_{p,\theta}B$ in the case where the function Ω has the special form

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right), \quad \omega \in \Phi^1_{\alpha,l}, \quad \alpha > 0. \tag{6}$$

Thus, $\omega(\cdot)$ is an arbitrary function (of one variable) of the type of a modulus of continuity of order l and $\omega \in \Phi^1_{\alpha,l}$. According to the previous definitions, it is clear that

$$\omega \in \Phi^1_{\alpha,l} \Longrightarrow \Omega \in \Phi^d_{\alpha,l}, \quad \alpha = (\underbrace{\alpha,\ldots,\alpha}_{d}).$$

Note that the set $\Phi^1_{\alpha,l}$, $l \in \mathbb{N}$, contains, e.g., the function

$$\omega(u) = \begin{cases} \dfrac{u^r}{\left(\log^+ \dfrac{1}{u}\right)^{\beta}}, & u > 0, \\ 0, & u = 0, \end{cases}$$

where $\log^+ \tau = \max\{1, \log \tau\}$, $0 < r < l$, $\beta \in \mathbb{R}$.

In what follows, we use the same notation for the unit ball in the space $S^{\Omega}_{p,\theta}B \cap L^{\circ}_p(\pi_d)$ as for the space $S^{\Omega}_{p,\theta}B$ itself, i.e.,

$$S^{\Omega}_{p,\theta}B := \{f \in S^{\Omega}_{p,\theta}B \cap L^{\circ}_p(\pi_d) \colon \|f\|_{S^{\Omega}_{p,\theta}B} \le 1\}.$$

1. Auxiliary Statements

We present several auxiliary statements that are used in the proof of the main results. First, we establish exact-order estimates for the best M-term trigonometric approximations of functions from the classes $S^{\Omega}_{\infty,\theta}B$.

For $f \in L_q(\pi_d)$, $1 \le q \le \infty$, we set

$$e_M(f)_q := \inf_{k^j,\, c_j} \left\| f(\cdot) - \sum_{j=1}^{M} c_j e^{i(k^j,\cdot)} \right\|_q, \tag{7}$$

where $\{k^j\}_{j=1}^M$ is a system of vectors $k^j = (k^j_1, \ldots, k^j_d)$ with integer-valued coordinates and c_j are arbitrary complex numbers. Quantity (7) is called the best M-term trigonometric approximation of the function f in the space L_q. If $F \subset L_q(\pi_d)$ is a certain functional class, then we denote

$$e_M(F)_q := \sup_{f \in F} e_M(f)_q. \tag{8}$$

For a function of one variable, the quantity $e_M(f)_2$ was introduced by Stechkin in [8] in the formulation of a criterion for the absolute convergence of trigonometric series. Later, the quantities $e_M(f)_q$ and $e_M(F)_q$ were investigated from the viewpoint of approximation. In particular, the behavior of quantity (8) for some classes of functions of many variables was studied in [9, 10] (see also the references therein). Also note that the behavior of the quantities of the best M-term approximation of the classes $S^{\Omega}_{p,\theta}B$ considered in the present paper was investigated in [11–13].

For $f \in L_q(\pi_d)$, $1 \le q \le \infty$, we introduce the quantity

$$e^{\perp}_M(f)_q := \inf_{k_j} \left\| f(\cdot) - \sum_{j=1}^{M} \hat{f}(k^j) e^{i(k^j,\cdot)} \right\|_q ,$$

which is called the best M-term orthogonal trigonometric approximation of the function f in the space L_q. If $F \subset L_q(\pi_d)$ is a certain functional class, then we set

$$e^{\perp}_M(F)_q := \sup_{f \in F} e^{\perp}_M(f)_q. \tag{9}$$

According to the definition, quantities (8) and (9) satisfy the relation

$$e_M(F)_q \le e^{\perp}_M(F)_q. \tag{10}$$

Theorem A (Littlewood–Paley theorem; see, e.g., [6, p. 65])**.** *Let $1 < p < \infty$ be given. Then there exist positive numbers C_7 and C_8 such that, for every function $f \in L_p(\pi_d)$, the following relations are true:*

$$C_7 \|f\|_p \le \left\| \left\{ \sum_s |\delta_s(f;\cdot)|^2 \right\}^{1/2} \right\|_p \le C_8 \|f\|_p. \tag{11}$$

Using inequalities (11), one can easily obtain the following relation (see, e.g., [14, p. 17]):

$$\|f\|_p \ll \left\{ \sum_s \|\delta_s(f;\cdot)\|_p^{p_0} \right\}^{1/p_0}, \tag{12}$$

where $p_0 = \min\{2; p\}$.

The following statement is true:

Theorem 1. *Suppose that $1 < q < \infty$, $1 \le \theta \le \infty$, and*

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j \right),$$

where

$$\omega \in \Phi^1_{\alpha,l}, \quad \alpha > \max\left\{0; \frac{1}{\theta} - \frac{1}{2}\right\}.$$

Then, for any sequence $M = (M_n)_{n=1}^{\infty}$ *of natural numbers such that* $M \asymp 2^n n^{d-1}$, *the following order equality is true:*

$$e_M(S^{\Omega}_{\infty,\theta}B)_q \asymp e^{\perp}_M(S^{\Omega}_{\infty,\theta}B)_q \asymp \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}. \tag{13}$$

Proof. For the determination of the upper bound for $e_M(S^{\Omega}_{\infty,\theta}B)_q$ we use inequality (10), the imbedding $S^{\Omega}_{\infty,\theta}B \subset S^{\Omega}_{p,\theta}B$, $1 \le p < \infty$, and the upper bound for $e^{\perp}_M(S^{\Omega}_{p,\theta}B)_q$, $1 < q \le p < \infty$, $p \ge 2$, established in [15]. As a result, we get

$$e_M(S^{\Omega}_{\infty,\theta}B)_q \le e^{\perp}_M(S^{\Omega}_{\infty,\theta}B)_q \le e^{\perp}_M(S^{\Omega}_{p,\theta}B)_q \ll \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}.$$

In [12], the following order relation was established:

$$e_M(S^{\Omega}_{\infty,\theta}B)_q \gg \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}, \qquad 1 < q \le 2, \quad 1 \le \theta \le \infty, \quad M \asymp 2^n n^{d-1}.$$

Using the monotonicity of the norm $\|\cdot\|_q$ with respect to the parameter $2 \le q < \infty$, we get

$$e_M(S^{\Omega}_{\infty,\theta}B)_q \ge e_M(S^{\Omega}_{\infty,\theta}B)_2 \gg \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}, \quad M \asymp 2^n n^{d-1}.$$

The theorem is proved.

Remark 1. Theorem 1 complements the estimates obtained in [12, 13].

2. Best Bilinear Approximations

We define the quantity that is investigated in this section.

Let $L_q(\pi_{2d})$, $q = (q_1, q_2)$, be the set of functions $f(x, y)$, $x, y \in \pi_d$, with the finite mixed norm

$$\|f(x,y)\|_{q_1,q_2} = \big\| \|f(\cdot,y)\|_{q_1} \big\|_{q_2},$$

where the norm is calculated first in the space $L_{q_1}(\pi_d)$ with respect to the variable $x \in \pi_d$ and then in the space $L_{q_2}(\pi_d)$ with respect to the variable $y \in \pi_d$. For $f \in L_q(\pi_{2d})$, we define the best bilinear approximation of order M as follows:

$$\tau_M(f)_{q_1,q_2} := \inf_{u_j(x),v_j(y)} \left\| f(x,y) - \sum_{j=1}^{M} u_j(x)v_j(y) \right\|_{q_1,q_2},$$

where $u_j \in L_{q_1}(\pi_d)$ and $v_j \in L_{q_2}(\pi_d)$.

If $F \subset L_q(\pi_{2d})$ is a class of functions, then we set

$$\tau_M(F)_{q_1,q_2} := \sup_{f \in F} \tau_M(f)_{q_1,q_2}. \tag{14}$$

The aim of this section is to establish exact-order estimates for the quantity

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,q_2} = \sup_{f\in S^{\Omega}_{p,\theta}B} \tau_M(f)_{q_1,q_2},$$

where the bilinear approximations $\tau_M(f)_{q_1,q_2}$ are considered for functions of the form $f(x-y)$, $x, y \in \pi_d$.

Note that the classic result for bilinear approximations belongs to Schmidt [17]. In [9, p. 10], Temlyakov formulated this result in a form more general than in [17].

Lemma A. *Suppose that* $\|K(x,y)\|_{2,2} < \infty$, K *is the integral operator with kernel* $K(x,y)$, K^* *is the operator adjoint to* K, *and* λ_j *is the nonincreasing sequence of eigenvalues of the operator* K^*K. *Then*

$$\inf_{u_i(x),v_i(y)} \left\| K(x,y) - \sum_{i=1}^{M} u_i(x)v_i(y) \right\|_{2,2} = \left(\sum_{j=M+1}^{\infty} \lambda_j \right)^{1/2}.$$

Quantity (14) with the classes $\mathrm{W}^r_{p,\alpha}$ and H^r_p taken as F was investigated by Temlyakov in [9, 18–20] (see also the references therein). The bilinear approximations of the Besov classes $B^r_{p,\theta}$ were studied by A. Romanyuk and V. Romanyuk in [16] and A. Romanyuk in [21].

We shall comment the obtained results by comparing them with estimates for the Kolmogorov widths.

Recall that the M-dimensional Kolmogorov width of a centrally symmetric set Φ of a Banach space $\mathcal{X}$ is defined as follows:

$$d_M(\Phi,\mathcal{X}) := \inf_{\mathcal{L}_M} \sup_{f\in\Phi} \inf_{u\in\mathcal{L}_M} \|f-u\|_{\mathcal{X}}, \tag{15}$$

where $\mathcal{L}_M$ is an arbitrary subspace of $\mathcal{X}$ of dimension M.

Let F be a certain class of functions and let $f(x)$ be a fixed function from F. By F_f we denote the set that consists of functions of the form $f(x-y)$ obtained from $f(x)$ by the displacement of its argument x by an arbitrary vector $y \in \pi_d$. Then the following equality is true (see, e.g., [9, p. 85]):

$$\tau_M(f(x-y))_{q_1,\infty} = d_M(F_f, L_{q_1}). \tag{16}$$

Thus, if the functional class F is invariant under the displacement of the argument of a function $f \in F$, then, according to (16), the values of $\tau_M(f(x-y))_{q_1,\infty}$ can be lower bounds for the Kolmogorov widths $d_M(F_f, L_{q_1})$.

The following statement is true:

Theorem 2. *Suppose that* $2 \le q_1 \le \infty$, $1 \le q_2, \theta \le \infty$, *and*

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right),$$

where

$$\omega \in \Phi^1_{\alpha,l}, \quad \alpha > \max\left\{0, \frac{1}{\theta} - \frac{1}{2}\right\}.$$

Then, for any sequence $M = (M_n)_{n=1}^{\infty}$ *of natural numbers such that* $M \asymp 2^n n^{d-1}$, *the following order equality is true:*

$$\tau_M(S^{\Omega}_{\infty,\theta}B)_{q_1,q_2} \asymp \omega(2^{-n}) n^{(d-1)(1/2-1/\theta)}. \tag{17}$$

Proof. The upper bounds in (17) can easily be obtained by using Theorem 1.

On the one hand, according to the estimate

$$e_M(S^{\Omega}_{\infty,\theta}B)_{q_1} \asymp \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}, \quad M \asymp 2^n n^{d-1},$$

for an arbitrary function f from the class $S^{\Omega}_{\infty,\theta}B$ one can find a set of vectors $k^1,\dots,k^M$, $k^j = (k^j_1,\dots,k^j_d)$, $k^j \in \mathbb{Z}^d$, $j=\overline{1,M}$, and numbers $c_1,\dots,c_M$ such that

$$\left\| f(x) - \sum_{j=1}^{M} c_j e^{i(k^j,x)} \right\|_{q_1} \ll \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}. \tag{18}$$

On the other hand, the left-hand side of (18) can be represented in the form

$$\begin{aligned}\left\| f(x) - \sum_{j=1}^{M} c_j e^{i(k^j,x)} \right\|_{q_1} &= \left\| f(x-y) - \sum_{j=1}^{M} c_j e^{i(k^j,x-y)} \right\|_{q_1,\infty} \\ &= \left\| f(x-y) - \sum_{j=1}^{M} c_j e^{i(k^j,x)} e^{-i(k^j,y)} \right\|_{q_1,\infty}. \end{aligned} \tag{19}$$

Using (18) and (19), we obtain

$$\left\| f(x-y) - \sum_{j=1}^{M} c_j e^{i(k^j,x)} e^{-i(k^j,y)} \right\|_{q_1,\infty} \ll \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}. \tag{20}$$

Setting $c_j e^{i(k^j,x)} = u_j(x)$ and $e^{-i(k^j,y)} = v_j(y)$ in (20), we establish the required upper bound for $\tau_M(S^{\Omega}_{\infty,\theta}B)_{q_1,\infty}$ and, hence, for $\tau_M(S^{\Omega}_{\infty,\theta}B)_{q_1,q_2}$.

Let us obtain the lower bound in (17).

Let M be an arbitrary natural number. We choose $n \in \mathbb{N}$ so that the number of elements of the set

$$Q_n = \bigcup_{\|s\|_1 = n} \rho(s)$$

satisfies the inequality $|Q_n| > 4M$. Also note that $|Q_n| \asymp 2^n n^{d-1}$.

Consider the functions

$$f_1(x) = C_9\omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta} \sum_{\|s\|_1=n} \prod_{j=1}^{d} R_{s_j}(x_j), \quad C_9 > 0, \quad 1 \le \theta < \infty,$$

and

$$f_2(x) = C_{10}\omega(2^{-n})2^{-n/2} \sum_{\|s\|_1=n} \prod_{j=1}^{d} R_{s_j}(x_j), \quad C_{10} > 0, \quad \theta = \infty,$$

where

$$R_{s_j}(x_j) = \sum_{l=2^{s_j-1}}^{2^{s_j}-1} \varepsilon_l e^{ilx}, \quad \varepsilon_l = \pm 1, \quad j = \overline{1, d},$$

are the Rudin–Shapiro polynomials, which, as is known, satisfy the order inequality $\|R_{s_j}\|_\infty \ll 2^{s_j/2}$ (see, e.g., [22, p. 155]).

We set

$$F_n(x) = \sum_{\|s\|_1 = n} \prod_{j=1}^{d} R_{s_j}(x_j).$$

Let us show that, for a certain value of the constant C_9, the function f_1 belongs to the class $S^{\Omega}_{\infty,\theta}B$, $1 \le \theta < \infty$, and the function f_2 with a certain constant C_{10} belongs to the class $S^{\Omega}_{\infty,\infty}B$. To this end, we first determine the norm of the function F_n in the corresponding spaces. For $1 \le \theta < \infty$, we have

$$\|F_n\|_{S^{\Omega}_{\infty,\theta}B} \asymp \left(\sum_s \omega^{-\theta}(2^{-\|s\|_1}) \|A_s(F_n, x)\|_\infty^\theta \right)^{1/\theta}$$

$$= \left(\sum_s \omega^{-\theta}(2^{-\|s\|_1}) \left\| A_s(x) * \sum_{\|s-s'\|_\infty \le 1} \delta_{s'}(F_n, x) \right\|_\infty^\theta \right)^{1/\theta}$$

$$\le \left(\sum_s \omega^{-\theta}(2^{-\|s\|_1}) \|A_s\|_1^\theta \left\| \sum_{\|s-s'\|_\infty \le 1} \delta_{s'}(F_n, x) \right\|_\infty^\theta \right)^{1/\theta}.$$

Taking into account that $\|A_s\|_1 \le 6$ (see, e.g., [14, p. 35]), we continue the estimate as follows:

$$\|F_n\|_{S^{\Omega}_{\infty,\theta}B} \ll \left(\sum_s \omega^{-\theta}(2^{-\|s\|_1}) \left\| \sum_{\|s-s'\|_\infty \le 1} \delta_{s'}(F_n, x) \right\|_\infty^\theta \right)^{1/\theta}$$

$$\le \left(\sum_s \omega^{-\theta}(2^{-\|s\|_1}) \left(\sum_{\|s-s'\|_\infty \le 1} \|\delta_{s'}(F_n, x)\|_\infty \right)^\theta \right)^{1/\theta}$$

$$= \left(\sum_s \omega^{-\theta}(2^{-\|s\|_1}) \left(\sum_{\|s-s'\|_\infty \le 1} \left\| \prod_{j=1}^{d} R_{s'_j}(x_j) \right\|_\infty \right)^\theta \right)^{1/\theta}$$

$$\ll \left(\sum_{\|s\|_1 \le n+d} \omega^{-\theta}(2^{-\|s\|_1}) \left(\sum_{\|s-s'\|_\infty \le 1} 2^{\frac{\|s'\|_1}{2}} \right)^{\theta} \right)^{1/\theta}$$

$$\ll \left(\sum_{\|s\|_1 \le n+d} \omega^{-\theta}(2^{-\|s\|_1}) 2^{\frac{\|s\|_1 \theta}{2}} \right)^{1/\theta}$$

$$= \left(\sum_{\|s\|_1 \le n+d} \frac{\omega^{-\theta}(2^{-\|s\|_1})}{2^{\alpha\theta\|s\|_1}} 2^{\frac{\|s\|_1 \theta}{2}} 2^{\alpha\theta\|s\|_1} \right)^{1/\theta}$$

$$\ll \frac{\omega^{-1}(2^{-(n+d)})}{2^{\alpha(n+d)}} \left(\sum_{\|s\|_1 \le n+d} 2^{\|s\|_1 \theta(1/2+\alpha)} \right)^{1/\theta}$$

$$\asymp \frac{\omega^{-1}(2^{-(n+d)})}{2^{\alpha(n+d)}} 2^{(n+d)(1/2+\alpha)} (n+d)^{(d-1)/\theta} \asymp \omega^{-1}(2^{-n}) 2^{n/2} n^{(d-1)/\theta}.$$

If $\theta = \infty$, then

$$\|F_n\|_{S^{\Omega}_{\infty,\infty}B} \ll \omega^{-1}(2^{-n}) 2^{n/2}.$$

This implies that, for certain values of the constants C_9 and C_{10}, the functions f_1 and f_2 belong to the classes $S^{\Omega}_{\infty,\theta}B$, $1 \le \theta < \infty$, and $S^{\Omega}_{\infty,\infty}B$, respectively.

We need the following auxiliary statement:

Lemma B [9, p. 98]**.** *Let a number M be given and let a number $n \in \mathbb{N}$ be such that the number of elements of the set*

$$Q_n = \bigcup_{\|s\|_1 = n} \rho(s)$$

satisfies the condition $|Q_n| > 4M$. Then, for an arbitrary function

$$g(x) = \sum_{k \in Q_n} \widehat{g}(k) e^{i(k,x)}$$

such that $|\widehat{g}(k)| = 1$, the following relation is true:

$$\inf_{u_j(x),\, \upsilon_j(y)} \left\| g(x-y) - \sum_{j=1}^{M} u_j(x)\, \upsilon_j(y) \right\|_{2,1} \gg M^{1/2}.$$

Since the function F_n satisfies the conditions of Lemma B, for $\tau_M(f_1(x-y))_{2,1}$ we get

$$\tau_M(f_1(x-y))_{2,1} \gg \omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta}\tau_M(F_n(x-y))_{2,1}$$

$$\gg M^{1/2}\omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta} \asymp \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}.$$

By analogy, for the function f_2 we obtain

$$\tau_M(f_2(x-y))_{2,1} \gg \omega(2^{-n})n^{(d-1)/2}.$$

The lower bound and the theorem are proved.

Remark 2. If $\omega(u) = u^r$, i.e.,

$$\Omega(t) = \prod_{j=1}^{d} t_j^r,$$

then, under certain restrictions on the parameter r, Theorems 1 and 2 yield the corresponding results for the classes $B^r_{\infty,\theta}$, which were established in [16].

Remark 3. Comparing Theorem 2 with the estimate for the Kolmogorov width $d_M(S^{\Omega}_{\infty,\theta}B, L_{q_1})$ established in [23], we obtain the order equalities

$$\tau_M(S^{\Omega}_{\infty,\theta}B)_{q_1,\infty} \asymp d_M(S^{\Omega}_{\infty,\theta}B, L_{q_1})$$

for $2 \le \theta < \infty$ and

$$\tau_M(S^{\Omega}_{\infty,\theta}B)_{q_1,\infty} \asymp d_M(S^{\Omega}_{\infty,\theta}B, L_{q_1})(\log^{d-1} M)^{(1/2-1/\theta)}$$

for $1 \le \theta < 2$.

Theorem 3. *Suppose that* $1 \le p \le 2 \le q_1 < \infty$, $1 \le q_2, \theta \le \infty$, *and*

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right),$$

where

$$\omega \in \Phi^1_{\alpha,l}, \quad \alpha > \frac{1}{p}, \quad l > \left[\frac{1}{p}\right].$$

Then, for any sequence $M = (M_n)_{n=1}^{\infty}$ *of natural numbers such that* $M \asymp 2^n n^{d-1}$, *the following order equality is true:*

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,q_2} \asymp \omega(2^{-n})2^{n(1/p-1/2)}n^{(d-1)(1/2-1/\theta)}. \tag{21}$$

Proof. As in the previous theorem, we establish the upper bounds by using the estimates for $e_M(S^{\Omega}_{p,\theta}B)$ obtained in [12, 13].

Further, we show that, for $1 \le p \le 2$, $\alpha > \dfrac{1}{p} - \dfrac{1}{2}$, and $1 \le \theta \le \infty$, one has the order inequality

$$\tau_M(S^{\Omega}_{p,\theta}B)_{2,1} \gg \omega(2^{-n})2^{n(1/p-1/2)}n^{(d-1)(1/2-1/\theta)}, \quad M \asymp 2^n n^{d-1}, \tag{22}$$

which yields the lower bound in (21).

Consider the case $p = 1$. For a given M, we choose a natural number n so that the number of elements of the set

$$Q_n = \bigcup_{\|s\|_1 = n} \rho(s)$$

satisfies the relations $|Q_n| > 2M$ and $|Q_n| \asymp M$.

Consider the functions

$$g_1(x) = C_{11} n^{-(d-1)/\theta} \sum_{n \le \|s\|_1 \le n+d} \omega(2^{-\|s\|_1}) \sum_{k \in \rho^+(s)} e^{i(k,x)}, \quad C_{11} > 0, \quad 1 \le \theta < \infty,$$

and

$$g_2(x) = C_{12} \sum_{n \le \|s\|_1 \le n+d} \omega(2^{-\|s\|_1}) \sum_{k \in \rho^+(s)} e^{i(k,x)}, \quad C_{12} > 0, \quad \theta = \infty,$$

where $\rho^+(s) = \left\{k : k = (k_1, \ldots, k_d),\ 2^{s_j - 1} \le k_j < 2^{s_j},\ j = \overline{1,d}\right\}$.

For the properly chosen constants C_{11} and C_{12}, the function g_1 belongs to the class $S^{\Omega}_{1,\theta}B$, $1 \le \theta < \infty$, and the function g_2 belongs to the class $S^{\Omega}_{1,\infty}B$. Indeed,

$$\|g_1\|_{S^{\Omega}_{1,\theta}B} \asymp \left(\sum_{n \le \|s\|_1 \le n+d} \omega^{-\theta}(2^{-\|s\|_1}) \|A_s(g_1, x)\|_1^{\theta} \right)^{1/\theta}$$

$$\ll n^{-(d-1)/\theta} \left(\sum_{n \le \|s\|_1 \le n+d} \omega^{-\theta}(2^{-\|s\|_1}) \omega^{\theta}(2^{-\|s\|_1}) \right)^{1/\theta}$$

$$= n^{-(d-1)/\theta} \left(\sum_{n \le \|s\|_1 \le n+d} 1 \right)^{1/\theta} \asymp n^{-(d-1)/\theta} n^{(d-1)/\theta} = 1,$$

$$\|g_2\|_{S^{\Omega}_{1,\infty}B} \asymp \sup_{n \le \|s\|_1 \le n+d} \frac{\|A_s(g_2, x)\|_1}{\omega(2^{-\|s\|_1})} \ll \sup_{n \le \|s\|_1 \le n+d} \frac{\omega(2^{-\|s\|_1})}{\omega(2^{-\|s\|_1})} = 1.$$

Using the function g as a kernel (here, for convenience, g is understood as g_1 for $1 \le \theta < \infty$ and g_2 for $\theta = \infty$), we consider the following integral operator $G\colon L_2 \longrightarrow L_2$:

$$(Gf)(x) = (2\pi)^{-d} \int\limits_{\pi_d} g(x-y) f(y) dy.$$

Let G^* be the operator adjoint to G and let λ_j be the eigenvalues of the operator G^*G arranged in the nonascending order. Since the eigenvalues λ_j coincide with the numbers $bn^{-\frac{2(d-1)}{\theta}}\omega^2(2^{-\|s\|_1})$, $b > 0$ (respectively, $b\omega^2(2^{-\|s\|_1})$ for $\theta = \infty$), by virtue of Lemma A we get

$$\inf_{u_i(x),v_i(y)} \|g_1(x-y) - \sum_{i=1}^{M} u_i(x)v_i(y)\|_{2,2}$$

$$= \left(\sum_{j \ge M+1} \lambda_j\right)^{1/2} \gg \left(\sum_{\|s\|_1 \ge n+1} n^{-\frac{2(d-1)}{\theta}} \omega^2(2^{-\|s\|_1})\right)^{1/2}$$

$$\gg n^{-(d-1)/\theta} \left(\sum_{\|s\|_1 \ge n+1} \omega^2(2^{-\|s\|_1}) \sum_{k \in \rho^+(s)} 1\right)^{1/2}$$

$$\asymp n^{-(d-1)/\theta} \left(\sum_{\|s\|_1 \ge n+1} \omega^2(2^{-\|s\|_1}) 2^{\|s\|_1}\right)^{1/2}$$

$$= n^{-(d-1)/\theta} \left(\sum_{\|s\|_1 \ge n+1} \frac{\omega^2(2^{-\|s\|_1})}{2^{-2\alpha\|s\|_1}} 2^{(1-2\alpha)\|s\|_1}\right)^{1/2}$$

$$\gg n^{-(d-1)/\theta} \frac{\omega(2^{-n})}{2^{-\alpha n}} \left(\sum_{\|s\|_1 \ge n+1} 2^{(1-2\alpha)\|s\|_1}\right)^{1/2}$$

$$\gg n^{-(d-1)/\theta} \frac{\omega(2^{-n})}{2^{-\alpha n}} 2^{(1-2\alpha)n/2} n^{(d-1)/2}$$

$$= \omega(2^{-n}) n^{(d-1)(1/2-1/\theta)} 2^{n/2}. \tag{23}$$

By analogy, for $\theta = \infty$ we obtain

$$\inf_{u_i(x),v_i(y)} \left\| g_2(x-y) - \sum_{i=1}^{M} u_i(x)v_i(y) \right\|_{2,2} \gg \omega(2^{-n}) 2^{n/2} n^{(d-1)/2}.$$

Further, let certain systems of functions $\{u_j(x)\}_{j=1}^M \in L_2(\pi_d)$ and $\{v_j(y)\}_{j=1}^M \in L_1(\pi_d)$ be given. Without loss of generality, we can assume that the functions $v_j(y)$, $j = \overline{1, M}$, are continuous. Let $u_g(x, y)$ denote the orthogonal projection of the function $g(x - y)$, for fixed y, to the subspace $U = \mathfrak{L}(\{u_j(x)\}_{j=1}^M)$ (the linear span of the functions $u_j(x)$, $j = \overline{1, M}$). We set

$$r(x, y) = g(x - y) - u_g(x, y).$$

Since the function $u_g(x, y)$ has the form

$$u_g(x, y) = \sum_{j=1}^{M} u_j(x)\varphi_j(y), \tag{24}$$

for an arbitrary $y \in \pi_d$ we obtain

$$\left\| g(\cdot - y) - \sum_{j=1}^{M} u_j(\cdot)v_j(y) \right\|_2 \geq \|r(\cdot, y)\|_2, \tag{25}$$

$$\|r(\cdot, y)\|_2 \leq \|g(\cdot - y)\|_2. \tag{26}$$

The function $r(x, y)$ satisfies the inequality

$$\|r(x, y)\|_{2,2}^2 \leq \|r(x, y)\|_{2,1} \|r(x, y)\|_{2,\infty}. \tag{27}$$

On the one hand, taking (24) into account, by analogy with (23) we get

$$\|r(x, y)\|_{2,2} = \|g(x - y) - u_g(x, y)\|_{2,2} \gg \omega(2^{-n})2^{n/2}n^{(d-1)(1/2-1/\theta)}. \tag{28}$$

On the other hand, we can estimate $\|r(x, y)\|_{2,\infty}$ from above. It follows from (26) that

$$\|r(x, y)\|_{2,\infty} \leq \|g\|_2. \tag{29}$$

Let us estimate $\|g\|_2$. Setting $g = g_1$, we obtain

$$\|g_1\|_2 = \left\| C_{11} n^{-(d-1)/\theta} \sum_{n \leq \|s\|_1 \leq n+d} \omega(2^{-\|s\|_1}) \sum_{k \in \rho^+(s)} e^{i(k,x)} \right\|_2$$

$$\asymp n^{-(d-1)/\theta} \left\| \sum_{n \leq \|s\|_1 \leq n+d} \omega(2^{-\|s\|_1}) \sum_{k \in \rho^+(s)} e^{i(k,x)} \right\|_2$$

$$\asymp n^{-(d-1)/\theta} \left(\sum_{n \leq \|s\|_1 \leq n+d} \omega^2(2^{-\|s\|_1}) \sum_{k \in \rho^+(s)} 1 \right)^{1/2}$$

$$\asymp n^{-(d-1)/\theta}\left(\sum_{n\le\|s\|_1\le n+d}\omega^2(2^{-\|s\|_1})2^{\|s\|_1}\right)^{1/2}$$

$$= n^{-(d-1)/\theta}\left(\sum_{n\le\|s\|_1\le n+d}\frac{\omega^2(2^{-\|s\|_1})}{2^{-2\alpha\|s\|_1}}2^{(1-2\alpha)\|s\|_1}\right)^{1/2}$$

$$\asymp n^{-(d-1)/\theta}\frac{\omega(2^{-n})}{2^{-\alpha n}}\left(\sum_{n\le\|s\|_1\le n+d}2^{(1-2\alpha)\|s\|_1}\right)^{1/2}$$

$$= n^{-(d-1)/\theta}\frac{\omega(2^{-n})}{2^{-\alpha n}}\left(\sum_{j=n}^{n+d}\sum_{\|s\|_1=j}2^{(1-2\alpha)\|s\|_1}\right)^{1/2}$$

$$\asymp n^{-(d-1)/\theta}\frac{\omega(2^{-n})}{2^{-\alpha n}}\left(\sum_{j=n}^{n+d}2^{(1-2\alpha)j}j^{d-1}\right)^{1/2}$$

$$\asymp n^{-(d-1)/\theta}\frac{\omega(2^{-n})}{2^{-\alpha n}}2^{(1-2\alpha)n/2}n^{(d-1)/2} = \omega(2^{-n})2^{n/2}n^{(d-1)(1/2-1/\theta)}.$$

Setting $g = g_2$, we get

$$\|g_2\|_2 = \left\|C_{12}\sum_{n\le\|s\|_1\le n+d}\omega(2^{-\|s\|_1})\sum_{k\in\rho^+(s)}e^{i(k,x)}\right\|_2 \asymp \omega(2^{-n})2^{n/2}n^{(d-1)/2}.$$

Using the estimates for $\|g_1\|_2$ and $\|g_2\|_2$ and inequality (29), for an arbitrary $1\le\theta\le\infty$ we obtain

$$\|r(x,y)\|_{2,\infty} \le \|g\|_2 \asymp \omega(2^{-n})2^{n/2}n^{(d-1)(1/2-1/\theta)}. \tag{30}$$

Relations (27)–(30) yield

$$\|r(x,y)\|_{2,1} \gg \omega(2^{-n})2^{n/2}n^{(d-1)(1/2-1/\theta)}.$$

Using inequality (25), we now obtain the required estimate for $p = 1$.

Consider the case $1 < p \le 2$. For a given M, we choose $n\in\mathbb{N}$ so that the number of elements of the set

$$Q_n = \bigcup_{\|s\|_1=n}\rho(s)$$

satisfies the relations $|Q_n| > 4M$ and $|Q_n| \asymp M$. Consider the functions

$$f_3(x) = C_{13}\omega(2^{-n})2^{-n(1-1/p)}n^{-(d-1)/\theta}d_n(x), \quad 1\le\theta<\infty,$$

and

$$f_4(x) = C_{14}\omega(2^{-n})2^{-n(1-1/p)}d_n(x), \quad \theta = \infty,$$

where

$$d_n(x) = \sum_{k\in Q_n} e^{i(k,x)}$$

and C_{13} and C_{14} are positive constants.

Since

$$\left\| \sum_{k_j=2^{s_j-1}}^{2^{s_j}-1} e^{ik_jx_j} \right\|_p \asymp 2^{s_j(1-1/p)}, \quad j=\overline{1,d},$$

we have

$$\|\delta_s(d_n,x)\|_p = \left\| \sum_{k\in\rho(s)} e^{i(k,x)} \right\|_p = \prod_{j=1}^d \left\| \sum_{k=2^{s_j-1}}^{2^{s_j}-1} e^{ik_jx_j} \right\|_p \asymp \prod_{j=1}^d 2^{s_j(1-1/p)} = 2^{\|s\|_1(1-1/p)}.$$

According to (2), for $1 \le \theta < \infty$ we get

$$\|f_3\|_{S^\Omega_{p,\theta}B} \asymp \left(\sum_{\|s\|_1=n} \omega^{-\theta}(2^{-\|s\|_1})\|\delta_s(f,x)\|_p^\theta \right)^{1/\theta}$$

$$\asymp \omega(2^{-n})2^{-n(1-1/p)}n^{-(d-1)/\theta} \left(\sum_{\|s\|_1=n} \omega^{-\theta}(2^{-\|s\|_1})\|\delta_s(d_n,x)\|_p^\theta \right)^{1/\theta}$$

$$\asymp \omega(2^{-n})2^{-n(1-1/p)}n^{-(d-1)/\theta} \left(\omega^{-\theta}(2^{-n}) \sum_{\|s\|_1=n} \|\delta_s(d_n,x)\|_p^\theta \right)^{1/\theta}$$

$$\asymp 2^{-n(1-1/p)}n^{-(d-1)/\theta} \left(\sum_{\|s\|_1=n} 2^{\theta\|s\|_1(1-1/p)} \right)^{1/\theta}$$

$$\asymp 2^{-n(1-1/p)}2^{n(1-1/p)}n^{-(d-1)/\theta} \left(\sum_{\|s\|_1=n} 1 \right)^{1/\theta} \ll 1.$$

For $\theta = \infty$, we have

$$\|f_4\|_{S^{\Omega}_{p,\infty}B} \asymp \sup_{\|s\|_1=n} \frac{\|\delta_s(f,x)\|_p}{\omega(2^{-\|s\|_1})} \asymp \omega(2^{-n})2^{-n(1-1/p)} \sup_{\|s\|_1=n} \frac{\|\delta_s(d_n,x)\|_p}{\omega(2^{-\|s\|_1})}$$

$$\asymp \omega(2^{-n})2^{-n(1-1/p)} \sup_{\|s\|_1=n} \frac{2^{\|s\|_1(1-1/p)}}{\omega(2^{-\|s\|_1})}$$

$$\asymp \omega(2^{-n})2^{-n(1-1/p)}\omega^{-1}(2^{-n})2^{n(1-1/p)} = 1.$$

Thus, the functions f_3 and f_4 belong to the classes $S^{\Omega}_{p,\theta}B$, $1 \le \theta < \infty$, and $S^{\Omega}_{p,\infty}B$, respectively, for certain values of the constants $C_{13}, C_{14} > 0$. Since the function d_n satisfies the conditions of Lemma B, for the functions f_3 and f_4 we get

$$\tau_M(f_3)_{2,1} \gg \omega(2^{-n})2^{-n(1-1/p)}n^{-(d-1)/\theta}M^{1/2}$$

$$\asymp \omega(2^{-n})2^{-n(1-1/p)}n^{-(d-1)/\theta}2^{n/2}n^{(d-1)/2}$$

$$= \omega(2^{-n})2^{n(1/p-1/2)}n^{(d-1)(1/2-1/\theta)},$$

$$\tau_M(f_4)_{2,1} \gg \omega(2^{-n})2^{-n(1-1/p)}M^{1/2} \asymp \omega(2^{-n})2^{n(1/p-1/2)}n^{(d-1)/2}.$$

The lower bound and the theorem are proved.

Remark 4. Comparing Theorem 3 with the estimate for the Kolmogorov width $d_M(S^{\Omega}_{p,\theta}B, L_{q_1})$ obtained in [3], we conclude that the following order equalities are true:

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,\infty} \asymp d_M(S^{\Omega}_{p,\theta}B, L_{q_1})$$

for $2 \le \theta < \infty$ and

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,\infty} \asymp d_M(S^{\Omega}_{p,\theta}B, L_{q_1})(\log^{d-1} M)^{(1/2-1/\theta)}$$

for $1 \le \theta < 2$.

Theorem 4. *Suppose that* $2 \le p < q_1 < \infty$, $1 \le q_2, \theta \le \infty$, *and*

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right),$$

where

$$\omega \in \Phi^1_{\alpha,l}, \quad \alpha > \frac{1}{2}.$$

Then, for any sequence $M = (M_n)_{n=1}^{\infty}$ *of natural numbers such that* $M \asymp 2^n n^{d-1}$, *the following estimate is true:*

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,q_2} \asymp \omega(2^{-n})n^{(d-1)(1/2-1/\theta)}.$$

Proof. As in the previous theorems, we obtain the upper bound by using the estimate for $e_M(S^{\Omega}_{p,\theta}B)_p$, $2 \le p < q_1 < \infty$, established in [13].

We now pass to the determination of the lower bounds. For a given M, we choose n so that $M \asymp 2^n n^{d-1}$ and $2^n n^{d-1} > 4M$.

Consider the functions

$$f_5(x) = C_{15}\omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta} \sum_{\|s\|_1=n} \prod_{j=1}^{d} R_{s_j}(x_j), \quad C_{15} > 0, \quad 1 \le \theta < \infty,$$

and

$$f_6(x) = C_{16}\omega(2^{-n})2^{-n/2} \sum_{\|s\|_1=n} \prod_{j=1}^{d} R_{s_j}(x_j), \quad C_{16} > 0, \quad \theta = \infty,$$

where

$$R_{s_j}(x_j) = \sum_{l=2^{s_j-1}}^{2^{s^j}-1} \varepsilon_l e^{ilx_j}, \quad \varepsilon_l = \pm 1, \quad j = \overline{1,d},$$

are the Rudin–Shapiro polynomials, for which, as indicated above, one has $\|R_{s_j}\|_\infty \ll 2^{s_j/2}$.

Let us show that, for a certain choice of the positive constants C_{15} and C_{16}, these functions belong to the classes $S^{\Omega}_{p,\theta}B$, $1 \le \theta < \infty$, and $S^{\Omega}_{p,\infty}B$, respectively. Since

$$\delta_s(f_5, x) = C_{15}\omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta} \prod_{j=1}^{d} R_{s_j}(x_j),$$

$$\delta_s(f_6, x) = C_{16}\omega(2^{-n})2^{-n/2} \prod_{j=1}^{d} R_{s_j}(x_j),$$

for $1 \le \theta < \infty$ we get

$$\|f_5\|_{S^{\Omega}_{p,\theta}B} \asymp \left(\sum_s \omega^{-\theta}(2^{-\|s\|_1})\|\delta_s(f_5,x)\|_p^\theta \right)^{1/\theta}$$

$$\asymp \omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta} \left(\sum_{\|s\|_1=n} \omega^{-\theta}(2^{-\|s\|_1}) \left\| \prod_{j=1}^{d} R_{s_j}(x_j) \right\|_p^\theta \right)^{1/\theta}$$

$$\ll \omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta}\left(\sum_{\|s\|_1=n}\omega^{-\theta}(2^{-\|s\|_1})2^{\frac{\|s\|_1\cdot\theta}{2}}\right)^{1/\theta}$$

$$\asymp \omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta}\omega^{-1}(2^{-\|s\|_1})2^{n/2}\left(\sum_{\|s\|_1=n}1\right)^{1/\theta}$$

$$\ll n^{-(d-1)/\theta}n^{(d-1)/\theta}=1.$$

For $\theta=\infty$, we obtain

$$\|f_6\|_{S^{\Omega}_{p,\infty}B}\asymp\sup_s\frac{\|\delta_s(f_6,x)\|_p}{\omega(2^{-\|s\|_1})}\asymp\omega(2^{-n})2^{-n/2}\sup_{\|s\|_1=n}\frac{\left\|\prod_{j=1}^d R_{s_j}(x_j)\right\|_p}{\omega(2^{-\|s\|_1})}$$

$$<\omega(2^{-n})2^{-n/2}\sup_{\|s\|_1=n}\frac{\left\|\prod_{j=1}^d R_{s_j}(x_j)\right\|_\infty}{\omega(2^{-\|s\|_1})}\ll\omega(2^{-n})2^{-n/2}\sup_{\|s\|_1=n}\frac{2^{\frac{\|s\|_1}{2}}}{\omega(2^{-\|s\|_1})}=1.$$

Taking into account that the function

$$\upsilon(x)=\sum_{\|s\|_1=n}\prod_{j=1}^d R_{s_j}(x_j)$$

satisfies the conditions of Lemma B, we get

$$\tau_M(f_5)_{2,1}\gg M^{1/2}\omega(2^{-n})2^{-n/2}n^{-(d-1)/\theta}\asymp\omega(2^{-n})n^{(d-1)(1/2-1/\theta)},$$

$$\tau_M(f_6)_{2,1}\gg M^{1/2}\omega(2^{-n})2^{-n/2}\asymp\omega(2^{-n})n^{(d-1)/2}.$$

The theorem is proved.

Remark 5. Comparing the estimate for the Kolmogorov width $d_M(S^{\Omega}_{p,\theta}B, L_{q_1})$ obtained in [3] with Theorem 4, we conclude that the following relations are true:

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,\infty}\asymp d_M(S^{\Omega}_{p,\theta}B, L_{q_1})$$

for $2\le\theta<\infty$ and

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,\infty}\asymp d_M(S^{\Omega}_{p,\theta}B, L_{q_1})(\log^{d-1}M)^{(1/2-1/\theta)}$$

for $1\le\theta<2$.

Theorem 5. *Suppose that* $2 \le q_1 \le p < \infty$, $1 \le q_2, \theta \le \infty$, *and*

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right), \quad \omega \in \Phi^1_{\alpha,l}, \quad \alpha > \max\left\{0; \frac{1}{\theta} - \frac{1}{2}\right\}.$$

Then, for any sequence $M = (M_n)_{n=1}^{\infty}$ *of natural numbers such that* $M \asymp 2^n n^{d-1}$, *the following order inequality is true:*

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,q_2} \asymp \omega(2^{-n}) n^{(d-1)(1/2-1/\theta)}.$$

Proof. The upper bound follows from the estimate for $e^{\perp}_M(S^{\Omega}_{p,\theta}B)_q$, $1 < q_1 \le p < \infty$, $p \ge 2$, obtained in [15]. The lower bound is established in the same way as in Theorem 4.

Remark 6. Comparing the estimate for the Kolmogorov width $d_M(S^{\Omega}_{p,\theta}B, L_{q_1})$ obtained in [24] with Theorem 5, we conclude that

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,\infty} \asymp d_M(S^{\Omega}_{p,\theta}B, L_{q_1})$$

for $\theta \ge 2$ and

$$\tau_M(S^{\Omega}_{p,\theta}B)_{q_1,\infty} \asymp d_M(S^{\Omega}_{p,\theta}B, L_{q_1})(\log^{d-1} M)^{(1/2-1/\theta)}$$

for $1 \le \theta < 2$.

Remark 7. If

$$\Omega(t) = \prod_{j=1}^{d} t_j^r,$$

then, under certain restrictions on the parameter r, Theorems 3–5 yield the known results for the classes $B^r_{p,\theta}$ established in [21].

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