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Tools of Lyapunov Functions for Qualitative Analysis of Time-Switched Linear System

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Abstract—**We investigate a switched system given by a set of linear systems. We offer tools of qualitative system analysis using multiple Lyapunov function method and obtain an estimation of system solution at final time moment depending of initial state. The estimation is obtained by composing of perturbations of separate subsystems on each time interval of switched system.**

I. Introduction

Nowadays many complex processes in real technical systems have changeable dynamics and very often lead to the study of hybrid and switched dynamical systems. They are of a major research interest due to their use as models in many applications in computer science and systems control. These systems have dynamics with several modes and their transitions from one mode to other occurs if solution reaches definite conditions. Such systems combine both the complex dynamics described by linear or nonlinear differential equations, and logical switches which cause transitions of technical systems for a various modes of operation [1],[3],[9],[18],[25]. They consist of a continuous-time and a discrete-time process, which interact with the logical law or decision-making process. Continuous and discrete subsystems can be represented by systems of differential and difference equations, respectively, where the logical laws or decision-making processes are expressed by a finite automaton or a discrete system. Switched systems are described by set of linear differential equations with logical elements of time-switch type. A time-switched controller can be used in a continuous system to achieve better performance benefits [17],[19]. These systems have numerous applications in control of mechanical systems, automotive industry, flight and air traffic control, switching power converters, process control, intelligent vehicle highway systems, robotics, etc. Examples include automation systems, repetitive, computer disks drives, automation, machine control, high-level flexible manufacturing systems, intelligent transportation systems, highway transport, maritime and air traffic control systems, control systems of modern spacecraft, electrical systems, chemical processes.

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Many problems associated with the investigation of switched systems are related to automatic programmatic control [11] and stability [2],[7],[12],[21],[27],[24]. In [22] challenging problems for stability and control in the hybrid dynamic system framework are posed.

One of the major problems in hybrid and switched dynamic systems is the establishing of their key property of stability, which is important in controller design. Stability may prove also critical for real-time systems, embedded systems, and hybrid systems in general that arise in computer science problems where verification tests are undecidable. Relaxing demands in the search for a stability proof may be necessary for a specific problem. For an overview of results on hybrid stability see [15], [8], [14], [20]. Some results assume arbitrary switching between locations, and it is possible to look for a Lyapunov function common to all locations [20]. Another stability criterion is multiple Lyapunov functions [3-5], [16] one, and piecewise quadratic Lyapunov function [10],[11]. Each location is assumed to have a Lyapunov function and in [9] the information about stability is given by available tools from non- smooth analysis to study the gradient information of the candidate Lyapunov functions. Paper [13] presents a novel model of a predictive control scheme that achieves input-to-state stabilization of hybrid systems. Input-to-state stability is guaranteed when an optimal solution of the optimization problem is attained. In [26] it is proved that the linear hybrid automata can be reduced to a linear discrete-time system with periodic coefficients. Moreover it is shown that the linear hybrid automata admit a periodic trajectory, and the theorem for asymptotic stability of the periodic trajectory is given. It is applied to prove the stability of the switched system.

The main purpose of the paper is to find the estimation of solution of switched systems with a time-switch controller described by linear subsystems. Using the estimations for separate subsystems, which are obtained with the method of Lyapunov functions, and using the continuity condition of solution at the switching moments, we get the estimation of the origin switched system solution at the final moment depending on initial state. Using the Lyapunov function of quadratic form with matrix exponential the estimations of solution are obtained. We get the final estimation by composing perturbations for the separate subsystems. Moreover we consider the scalar case of such systems.

II. PRELIMINARIES

A. Switched System in General Form.

We consider a switched system given by a set of linear subsystems which evolve over finite time intervals, $T_i = [t_{i-1}, t_i]$, $i = \overline{1, N}$. The moments of time $t_1, t_2, ..., t_i, ..., t_N$ are called *switching moments*. Such switched system can be written as a family of systems which are described by

$$\dot{x}(t) = A_i x(t), \; i = \overline{1, N}, \; t \geq 0 \tag{1}$$

where $x(t) \in R^n$, A_i - constant matrices.

For solutions $x = x(x_0, t)$ the continuity condition

$$\lim_{s \to +0} x(t_i - s) = \lim_{s \to +0} x(t_i + s), \, i = \overline{1, N} \tag{2}$$

holds at switching moments.

B. Switched System Given by Scalar Equations

A particular case we consider the switched system functions over the time interval $T_i = [t_{i-1}, t_i]$, $i = \overline{1, 3}$ given by three linear scalar systems

$$\begin{aligned} \dot{x}(t) &= \lambda_1^i x(t) + b^i y(t) \\ \dot{y}(t) &= \lambda_2^i y(t), \end{aligned} \tag{3}$$

where $x(t) \in R^n$, $\lambda_1^i, \lambda_2^i, b^i$ - are constant numbers.

For solution $x(t)$ of system (3) the continuity condition (2) holds at switching moments too.

C. Statement of the problem

The main goal of the paper is to find the estimation of the solution of switched systems in general form (1) and in scalar form (3) depending on the initial state. We calculate the estimations of solutions $|x(t_N)|$ at the final moment $t = t_N$. The switched systems presented by set of subsystems which are linear differential equations with constant coefficients are considered. Each of the subsystems describes the evolution dynamic on the finite time interval, $t_{i-1} \leq t < t_i$ $i = \overline{1, N}$. Subsystems can be either stable or unstable.

We assume, that initial state of switched system (1) and system (3) are satisfied to

$$|x(0)| < \delta .$$

It means that all solutions of the system $x(0)$ are in δ–neighborhood of equilibrium at the initial time moment $t = t_0$.

On the separate time intervals the subsystem with matrices A_i are described by the systems of linear differential equations, having the continuous dependence of solutions on initial conditions. Therefore for arbitrary $\delta > 0$ all solutions starting from δ–neighborhood will not leave $\varepsilon(\delta)$–neighborhood. And contrary, for arbitrary $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $|x(t_N)| < \varepsilon$ if $|x(0)| < \delta$. The paper is devoted to the problems of estimation of solutions on the finite time intervals, so the calculation of these values. As subsystems are linear, the Lyapunov functions of quadratic form are used.

The main goal is to find the estimation of systems (1) and (3) depending of initial state. We compute the estimations of solutions $|x(t_N)|$ at the final moment $t = t_N$.

At first we obtain the estimations of solutions of separate subsystems using the method of Lyapunov functions. Next stage is following: taking into account the condition (2) of continuity between subsystems at the switching moments, we obtain the estimation at the final moment. We get the estimation by composition of perturbations for the separate subsystems.

We use the following vector norm:

$$|x(t)| = \sqrt{\sum_{i=1}^{N} x_i^2(t)}$$

Paper was organizing as follows. At first we deal with particular subsystems and obtain the estimations. Next we collect all particular results and compute the final estimation of initial switched system. This investigation is made using two tools: quadratic multiple Lyapunov functions and coinciding multiple Lyapunov functions.

III. TOOLS OF QUADRATIC MULTIPLE LYAPUNOV FUNCTIONS

A. Estimation of Solution of General Switched System.

It is known that for the linear system with constant coefficients

$$\dot{x}(t) = Ax(t), \, x(t) \in R^n, \, t \geq 0$$

a general solution is $x(x_0, t) = e^{At} x_0$ where matrix exponential has the form

$$e^{At} = I + A\frac{t}{1!} + A^2 \frac{t^2}{2!} + \cdots + A^k \frac{t^k}{k!} + \cdots$$

And the solution of the system can be estimated with the use the Lyapunov function, which has a quadratic form [2]

$$V(x,t) = x_0^T x_0 = \left(e^{-At} x\right)^T \left(e^{-At} x\right),$$

or

$$V(x,t) = x^T H(t) x, \;\; H(t) = e^{-A^T t} e^{-At} .$$

This form of Lyapunov function describes the evolution of process the most exactly. The space $V(x,t) < \alpha$ is a middle of ellipse which goes out from the neighborhood $|x_0| < \alpha$ but changes along the solutions $x = x(x_0, t)$. The multiple

Lyapunov functions of the switched system are shown on Figure 1.

Theorem 1. Let the initial state of the switched system (1) satisfied the condition $|x(0)|<\delta$. Then at $t=t_N$ inequality

$$|x(t_N-0)|<\frac{\delta}{\sqrt{\prod_{i=1}^{N}\lambda_{\min}[H_i(t_i)]}}, \qquad (4)$$

holds, where $H_i(t_i)=e^{-A_i^T(t_i-t_{i-1})}e^{-A_i(t_i-t_{i-1})}$, $i=\overline{1,N}$, $t_0=0$.

Proof. We consider the first time interval of dynamic evolution of system (1):

$$\dot{x}(t)=A_1x(t),\ \ x(t)\in R^n,\ \ t_0\le t<t_1,\ \ t_0=0. \qquad (5)$$

We select the Lyapunov function in such form

$$V(x,t)=x^TH_1(t)x,\ \ H_1(t)=e^{-A_1^Tt}e^{-A_1t}.$$

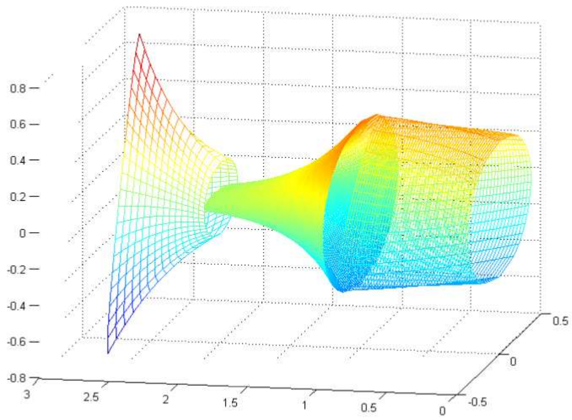

Fig. 1. Quadratic Multiple Lyapunov Functions

For the Lyapunov function the bilateral inequality

$$\lambda_{\min}[H_1(t)]|x(t)|^2\le V_1(x(t),t)\le\lambda_{\max}[H_1(t)]|x(t)|^2$$

is hold.

A complete derivative of function $V_1(x,t)$ for the subsystem (5) yields

$$\frac{d}{dt}V_1(x(t),t)=\dot{x}^T(t)H_1(t)x(t)+ \\ +x^T(t)\frac{d}{dt}H_1(t)x(t)+x^T(t)H_1(t)\dot{x}(t)\equiv 0.$$

Therefore, at arbitrary t: $0\le t<t_1$ Lyapunov function along the trajectories is constant i.e.

$$V_1(x(t),t)\equiv V_1(x(0),0)=\text{const},\ \ 0\le t<t_1.$$

Obtaining

$$\lambda_{\min}\left[H_1(t_1-0)\right]\left|x(t_1-0)\right|^2\le V_1(x(t_1-0),t_1-0)= \\ =V_1(x(0),0)\le\lambda_{\max}\left[H_1(0)\right]\left|x(0)\right|^2=\left|x(0)\right|^2$$

And for the first time interval we get inequality

$$|x_1(t_1-0)|\le\frac{|x(0)|}{\sqrt{\lambda_{\min}[H_1(t_1)]}}.$$

We consider the second time interval, and the subsystem

$$\dot{x}(t)=A_2x(t),\ \ x(t)\in R^n,\ \ t_1\le t<t_2. \qquad (6)$$

The Lyapunov function on this time interval has a form

$$V_2(x,t)=x^TH_2(t)x,\ \ H_2(t)=e^{-A_2^T(t-t_1)}e^{-A_2(t-t_1)}.$$

The bilateral inequality

$$\lambda_{\min}[H_2(t)]|x(t)|^2\le V_2(x(t),t)\le\lambda_{\max}[H_2(t)]|x(t)|^2.$$

is hold.

The complete derivative of function $V_2(x,t)$ for subsystem (6) is

$$\frac{d}{dt}V_2(x(t),t)\equiv 0,\ \ t_1\le t<t_2,$$

And

$$V_2(x(t),t)\equiv V_2(x\ (t_1+0),t_1+0)=\text{const},\ \ t_1\le t<t_2.$$

From here

$$\lambda_{\min}[H_2(t_2-0)]|x(t_2-0)|^2\le V_2(x(t_2-0),t_2-0)= \\ =V_2(x(t_1+0),t_1+0)\le \\ \le\lambda_{\max}[H_2(t_1+0)]|x(t_1+0)|^2=|x(t_1+0)|^2.$$

And for the second time interval we get

$$\left|x(t_2-0)\right|\le\frac{\left|x(t_1+0)\right|}{\sqrt{\lambda_{\min}\left[H_2(t_2)\right]}}.$$

So the continuity condition (2) takes place then we obtain

$$|x_2(t_2-0)|\le\frac{|x(0)|}{\sqrt{\lambda_{\min}[H_2(t_2)]\lambda_{\min}[H_1(t_1)]}}$$

holds at the switched moment.

Continuing the process farther, we get

$$|x_N(t_N-0)|\le\frac{|x(0)|}{\sqrt{\prod_{i=1}^{N}\lambda_{\min}[H_i(t_i)]}}, \qquad (7)$$

concluding the proof of the Theorem 1.

B. *Obtaining the Estimation of Scalar Switched System*

Consider the linear scalar system

$$\dot{x}(t)=\lambda_1x(t)+by(t)$$
$$\dot{y}(t)=\lambda_2y(t),\qquad \lambda_1\ne\lambda_2.$$

The eigenvector $(\alpha_1,\beta_1)=(1,0)$ corresponds to eigenvalue $\lambda=\lambda_1$, and eigenvalue $(\alpha_2,\beta_2)=(b,\lambda_2-\lambda_1)$ corresponds to eigenvalue $\lambda=\lambda_2$. Therefore the fundamental matrix of solution is

$$X(t)=\begin{bmatrix} e^{\lambda_1t} & be^{\lambda_2t} \\ 0 & (\lambda_2-\lambda_1)e^{\lambda_2t}\end{bmatrix}.$$

So

$$X(0)=\begin{bmatrix} 1 & b \\ 0 & (\lambda_2-\lambda_1)\end{bmatrix},$$

then

$$e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} e^{\lambda_1 t} & be^{\lambda_2 t} \\ 0 & (\lambda_2-\lambda_1)e^{\lambda_2 t} \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1 & b \\ 0 & (\lambda_2-\lambda_1) \end{bmatrix}^{-1} = \begin{bmatrix} e^{\lambda_1 t} & \dfrac{b}{\lambda_2-\lambda_1}\left(e^{\lambda_2 t}-e^{\lambda_1 t}\right) \\ 0 & e^{\lambda_2 t} \end{bmatrix}.$$

The matrix $H(t) = e^{-A^T t}e^{-At}$ of Lyapunov function $V(x,t) = x^T H(t)x$ has a form

$$H(t) = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix}$$

where coefficients h_{11}, h_{12}, h_{22} which are depended of λ_1, λ_2 and can be computed.

And we obtain a Lyapunov function

$$V(x,y,t) = h_{11}(t)x^2 + 2h_{12}(t)xy + h_{22}(t)y^2 , \qquad (8)$$

The eigenvalues of matrix $H(t)$ can be obtained from the characteristic equation

$$\det\left[H(t) - \kappa I\right] = \det\begin{vmatrix} h_{11}(t)-\lambda & h_{12}(t) \\ h_{12}(t) & h_{22}(t)-\lambda \end{vmatrix} = 0 .$$

We get

$$\lambda_{12}(H(t)) = \frac{1}{2}\left\{\left[h_{11}(t)+h_{22}(t)\right] \pm \sqrt{\left[h_{11}(t)-h_{22}(t)\right]^2 + 4h_{12}^2(t)}\right\}.$$

Example 1. Consider the switched system on three time intervals: $0 \le t < 1$, $1 \le t < 2$ and $2 \le t \le 3$. We compute the estimation $\left|x(t_3)\right|$ at the time moment $t = 3$.

The initial state of the solution of system (3) at $t = 0$ is equal δ_0 : $\left|x(0)\right| = \delta_0$.

1) Over the time interval $0 \le t < 1$ the system coefficients are $\lambda_1 = -1$, $\lambda_2 = 0$, $b = 1$. Therefore the system has a form

$$\begin{aligned} \dot{x}(t) &= -x(t) + y(t), \\ \dot{y}(t) &= 0. \end{aligned} \qquad (9)$$

Lyapunov function (8) has a form:

$$V(t,x,y) = e^{2t}x^2 + 2e^t(1-e^t)xy + \left[(1-e^t)^2 + 1\right]y^2 .$$

The eigenvalues of matrix $H(t)$ at $t = t_1$ are

$$\lambda_{1,2}(H(t)) =$$

$$= \frac{1}{2}\left[(1-e^t)^2 + (1-e^{2t})\right] \pm \frac{1}{2}(1-e^t)\sqrt{((1-e^t)+1+e^t)^2+4} ,$$

And compute $\lambda_{\max}\left[H(t_1)\right] = 10{,}58$, $\lambda_{\min}\left[H(t_1)\right] = 0{,}54$.

The estimation of solution at $t = 1$:

$$\left|x(t_1)\right| = \delta_1 = \frac{\delta_0}{\sqrt{\lambda_{\min}\left[H(t_1)\right]}} = \frac{\delta_0}{\sqrt{0{,}54}} . \qquad (10)$$

2) Over the time interval $1 \le t < 2$ the system has a form:

$$\begin{aligned} \dot{x}(t) &= -x(t) + y(t) \\ \dot{y}(t) &= -2y(t). \end{aligned} \qquad (11)$$

Lyapunov function (8) yields:

$$V(t,x,y) = (e^{2t})x^2 + 2e^t(e^{2t}-e^t)xy + ((e^{2t}-e^t)^2 + e^{4t})y^2 .$$

The estimation of solution at $t = 2$ over the interval $1 \le t < 2$ we get taking into account the initial state δ_1 which has been computed in (10):

$$\left|x(t_2)\right| = \delta_2 = \frac{\delta_0}{\sqrt{\lambda_{\min}\left[H(t_1)\right]\lambda_{\min}\left[H(t_2)\right]}} =$$

$$= \frac{\delta_0}{\sqrt{0{,}54 \cdot 24{,}5}} = \frac{\delta_0}{\sqrt{13{,}23}}.$$

3) On the interval $2 \le t < 3$ system has a form:

$$\begin{aligned} \dot{x}(t) &= -2x(t) + y(t) \\ \dot{y}(t) &= 2y(t). \end{aligned} \qquad (12)$$

Lyapunov function:

$$V(t,x,y) = (e^{4t})x^2 + \frac{1}{2}e^{2t}(e^{-2t}-e^{2t})xy +$$

$$+\left[\frac{1}{16}(e^{-2t}-e^{2t})^2 + e^{-4t}\right]y^2 .$$

The estimation of solution at $t = 3$:

$$\left|x(t_3)\right| = \frac{\delta_0}{\sqrt{\lambda_{\min}\left[H(t_1)\right]\lambda_{\min}\left[H(t_2)\right]\lambda_{\min}\left[H(t_3)\right]}} =$$

$$= \frac{\delta_0}{\sqrt{0{,}54 \cdot 24{,}5 \cdot 0.05}} = \frac{\delta_0}{\sqrt{0{,}6615}}.$$

The multiple Lyapunov functions constructed for this switched system are given on Figure 1. One can see three Lyapunov functions constructed for three time intervals. The initial condition is assumed $\left|x(0)\right| = \delta_0 = 1$. The Lyapunov functions of each time interval are expanded or narrowed depending of each system stability. The estimations of solutions $\left|x(t_i)\right|, t = 1,2,3$ are computed at each time moment.

C. *Tools of Coinciding Multiple Lyapunov functions*

Proving of the Theorem 1 at the time moments $t = t_i$, $i = \overline{1,N}$ we have too strong restriction. The first restriction is the requirement for the curve level to $(i-1)$ - Lyapunov function $V_{i-1}(x,t)$ on $(i-1)$-step, which have the ellipse form on the ending of $(i-1)$-step. At the switching time moments this ellipse is inside of the curve level of i - Lyapunov function $V_i(x,t)$, which is a circle on the starting of i - step.

In this case we impose a weaker condition, when the curve level of i - Lyapunov function on $(i-1)$ - step is an ellipse which coincides with the ellipse of previous Lyapunov function at the moment of switching (see Figure 2).

Theorem 2. Let the initial state of switched system (1) satisfies the condition $|x(0)|<\delta$. Then at $t=t_N$ the inequality

$$|x(t_N-0)|<\frac{|\delta|}{\sqrt{\lambda_{\min}[H_N(t_N)]}}, \qquad (13)$$

holds where

$$H_N(t_N)=\prod_{i=N}^{1}e^{-A_i^T(t_i-t_{i-1})}\prod_{j=1}^{N}e^{-A_j(t_j-t_{j-1})},\ t_0=0.$$

The proof is made analogically to the previous Theorem, by keeping in mind that for the first time interval $0\le t<t_1$ Lyapunov function is

$$V_1(x,t)=x^T H_1(t)x,\ H_1(t)=e^{-A_1^T t}e^{-A_1 t}.$$

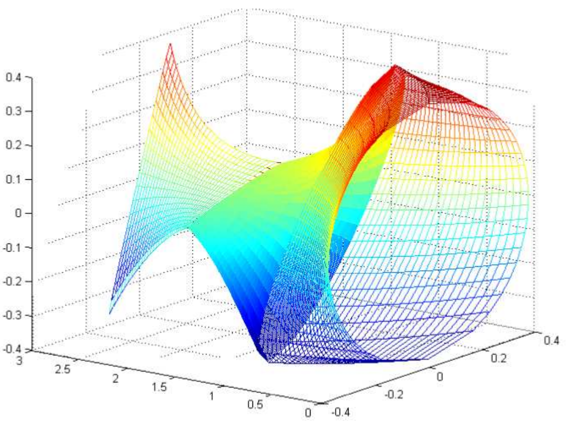

Fig. 2. Coinsiding Multiple Lyapunov Functions

or

$$V_1(x,t)=(e^{-A_1 t}x)^T e^{-A_1 t}x=x^T(e^{-A_1^T t}e^{-A_1 t})x,$$

and for the second time interval we select the „deformed" Lyapunov function

$$V_2(x,t)=(C_2e^{-A_2 t}x)^T C_2e^{-A_2 t}x=x^T(e^{-A_2^T t}C_2^T C_2e^{-A_2 t})x,$$

where C_2 is a non-zero matrix with constant coefficients. A condition coinciding at the switched moment $t=t_1$ determines the matrix C_2 is following

$$e^{-A_1 t_1}=C_2e^{-A_2 t_1}.$$

From here we obtain

$$C_2=e^{-A_1 t_1}e^{A_2 t_1}. \qquad (14)$$

At the second interval $t_1\le t<t_2$ the Lyapunov function

$$V_2(x,t)=x^T(e^{-A_2^T t}e^{A_2^T t_1}e^{-A_1^T t_1}e^{-A_1 t_1}e^{A_2 t_1}e^{-A_2 t})x=x^T H_2(t)x,$$

where

$$H_2(t)=e^{-A_2^T(t-t_1)}e^{-A_1^T t_1}e^{-A_1 t_1}e^{-A_2(t-t_1)}.$$

Example 2. We estimate the same switched system considered in Example 1 using the coinciding Lyapunov functions. Therefore we have $|x(0)|=\delta_0$ and three time intervals: $0\le t<1$, $1\le t<2$ and $2\le t\le 3$. We compute the estimation $|x(t_3)|$ at the time moment $t=3$.

1) At the time interval $0\le t<1$ the system has a form (9), and the estimation yields (10).

2) At $1\le t<2$ for the system (11) "deformed" Lyapunov function is

$$V_2(x,t)=x^T(e^{-A_2^T t}e^{A_2^T t_1}e^{-A_1^T t_1}e^{-A_1 t_1}e^{A_2 t_1}e^{-A_2 t})x=x^T H_2(t)x,$$

where

$$H_2(t)=(e^{-A_2^T t}e^{A_2^T t_1}e^{-A_1^T t_1}e^{-A_1 t_1}e^{A_2 t_1}e^{-A_2 t}).$$

Compute:

$$e^{A_2^T t_1}=\begin{bmatrix} e^{-1} & 0\\ \left(e^{-1}-e^{-2}\right) & e^{-2}\end{bmatrix},\quad e^{A_2 t_1}=\begin{bmatrix} e^{-1} & \left(e^{-1}-e^{-2}\right)\\ 0 & e^{-2}\end{bmatrix},$$

$$e^{-A_2 t}=\begin{bmatrix} e^{t} & \left(e^{t}-e^{2t}\right)\\ 0 & e^{2t}\end{bmatrix},$$

$$e^{-A_2^T t}=\begin{bmatrix} e^{t} & 0\\ \left(e^{t}-e^{2t}\right) & e^{2t}\end{bmatrix},\quad e^{-A_1 t_1}=\begin{bmatrix} e & (1-e)\\ 0 & 1\end{bmatrix},$$

$$e^{-A_1^T t_1}=\begin{bmatrix} e & 0\\ (1-e) & 1\end{bmatrix}.$$

From here we obtain matrix $H_2(t)$

$$H_2(t)=$$
$$=\begin{bmatrix} e^{2t}, & e^{2t}-2e^{3t-1}+e^{3t-2}\\ e^{2t}-2e^{3t-1}+e^{3t-2}, & e^{2t}+2e^{4t-4}+4e^{4t-2}-4e^{3t-1}-4e^{4t-3}\end{bmatrix}$$

.

Lyapunov function:

$$V(t,x,y)=e^{2t}x^2+2\left(e^{2t}-2e^{3t-1}+e^{3t-2}\right)xy+ \\ +\left(e^{2t}+2e^{4t-4}+4e^{4t-2}-4e^{3t-1}-4e^{4t-3}\right)y^2.$$

At $t=2$ we get

$$H_2(t_2)=\begin{bmatrix} e^4, & 2e^4-2e^5\\ 2e^4-2e^5, & 3e^4+4e^6-8e^5\end{bmatrix}.$$

And compute $\lambda_{\min}[H(t_2)]=$ 13,15.

Obtain the estimation:

$$|x(t_2)|=\delta_2=\frac{\delta_0}{\sqrt{\lambda_{\min}[H(t_2)]}}=\frac{\delta_0}{\sqrt{13,15}}.$$

3) For system (12) we obtain the estimation at $2\le t<3$

$$|x(t_3)|=\delta_3=\frac{\delta_0}{\sqrt{\lambda_{\min}[H(t_3)]}}=\frac{\delta_0}{\sqrt{0,67}}.$$

The dynamic of the switched system and coinciding Lyapunov functions are shown on Figure 2. The Lyapunov functions are constructed and estimations of solutions $|x(t_i)|, t=1,2,3$ are computed for three time intervals. The Lyapunov functions of each time interval are expanded or narrowed depending of stability of each system analogically to Example 1. But in contrast to Example 1 the ellipses of curve levels of the Lyapunov functions are coincided at the switched moments.

IV. Conclusion

We obtained the estimation of solution of switched system given by linear systems using the Lyapunov function method. The Lyapunov functions were selected in the quadratic form with matrix exponential. This investigation is made using two tools: quadratic multiple Lyapunov functions and coinciding multiple Lyapunov functions.

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