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APPROXIMATION OF (ψ, β)-DIFFERENTIABLE FUNCTIONS OF LOW SMOOTHNESS BY BIHARMONIC POISSON INTEGRALS

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We solve the Kolmogorov–Nikol'skii problem for biharmonic Poisson integrals on the classes of (ψ, β)-differentiable periodic functions of low smoothness in the uniform metric.

1. Statement of the Problem and Historical Notes

Let L_1 be the space of 2π-periodic summable functions with the norm

$$\|f\|_{L_1} = \|f\|_1 = \int_{-\pi}^{\pi} |f(t)|dt,$$

let L_∞ be the space of 2π-periodic, measurable, essentially bounded functions with the norm

$$\|f\|_{L_\infty} = \|f\|_\infty = \operatorname*{ess\,sup}_t |f(t)|,$$

and let C be the space of 2π-periodic continuous functions with the norm

$$\|f\|_C = \max_t |f(t)|.$$

Assume that $U(\rho; x)$ is a biharmonic function in the unit disk $|\rho e^{ix}| < 1$, i.e., it is a solution of the equation

$$\Delta^2 U(\rho; x) = 0, \tag{1}$$

where $\Delta^2 U(\rho; x) = \Delta(\Delta U(\rho; x))$ and

$$\Delta = \frac{1}{\rho^2}\frac{\partial^2}{\partial x^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right)$$

is the Laplace operator.

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Let $B(\rho; f; x)$ denote the solution of Eq. (1) with the boundary conditions

$$\left.\frac{\partial U(\rho;x)}{\partial x}\right|_{\rho=1} = 0, \qquad U(\rho;x)|_{\rho=1} = f(x),$$

where $f(x)$ is a summable 2π-periodic function.

It was shown in [1, p. 256] that the function $B(\rho; f; x)$, which is called the biharmonic Poisson integral of the function $f(\cdot)$, admits the following representation:

$$B(\rho; f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left\{\frac{1}{2} + \sum_{k=1}^{\infty}\left[1 + \frac{k}{2}\left(1-\rho^2\right)\right]\rho^k \cos kt\right\}dt.$$

We use the function

$$B_\delta(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left\{\frac{1}{2} + \sum_{k=1}^{\infty}\left[1 + \frac{k}{2}\left(1-e^{-2/\delta}\right)\right]e^{-k/\delta}\cos kt\right\}dt, \quad \delta > 0, \quad \rho = e^{-1/\delta},$$

as the basis of a linear method for approximation of functions from the classes $C^{\psi}_{\beta,\infty}$ introduced by Stepanets [2] as follows:

Let $\psi(k)$ be an arbitrary fixed function of a natural argument, let β be a fixed real number, and let $a_k(f)$ and $b_k(f)$ be the Fourier coefficients of a function f. If

$$\sum_{k=1}^{\infty}\frac{1}{\psi(k)}\left(a_k(f)\cos\left(kx + \frac{\pi\beta}{2}\right) + b_k(f)\sin\left(kx + \frac{\pi\beta}{2}\right)\right)$$

is the Fourier series of a function $\varphi \in L_1$, then $\varphi(\cdot)$ is called the (ψ,β)-derivative of f and is denoted by $f^{\psi}_{\beta}(\cdot)$. The class of continuous functions $f(\cdot)$ for which $\|f^{\psi}_{\beta}\|_\infty \le 1$ is denoted by $C^{\psi}_{\beta,\infty}$. Note that, for $\psi(k) = k^{-r}$, $r > 0$, the classes $C^{\psi}_{\beta,\infty}$ coincide with the classes $W^r_{\beta,\infty}$, and $f^{\psi}_{\beta} = f^{(r)}_{\beta}$ is the Weyl–Nagy (r,β)-derivative (see [3] and [4, p. 24]). If, in addition, one has $\beta = r$, $r \in \mathbb{N}$, then f^{ψ}_{β} is the rth-order derivative of f, and $C^{\psi}_{\beta,\infty}$ are the well-known Sobolev classes W^r_∞.

Following Stepanets (see [4, p. 93] and [5 p. 195]), we denote by $\mathfrak{M}$ the set of positive, continuous, convex-downward functions $\psi(u)$, $u \ge 1$, such that

$$\lim_{u\to\infty}\psi(u) = 0.$$

Let $\mathfrak{M}'$ denote the subset of functions $\psi \in \mathfrak{M}$ that satisfy the condition

$$\int_1^{\infty}\frac{\psi(t)}{t}dt < \infty.$$

We also consider the following subset of $\mathfrak{M}$ (see, e.g., [5, p. 160]):

$$\mathfrak{M}_0 = \left\{\psi \in \mathfrak{M}\colon\ 0 < \frac{t}{\eta(t) - t} \le K \ \ \forall t \ge 1\right\},$$

where

$$\eta(t) = \eta(\psi; t) = \psi^{-1}\left(\frac{1}{2}\psi(t)\right),$$

ψ^{-1} is the function inverse to $\psi,$ and K is a constant that may depend on $\psi.$ Also denote $\mathfrak{M}_0' = \mathfrak{M}_0 \cap \mathfrak{M}'.$

In the present work, we study the asymptotic behavior of the quantity

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; B_\delta\right)_C = \sup_{f \in C^{\psi}_{\beta,\infty}} \|f(\cdot) - B_\delta(f;\cdot)\|_C = \sup_{f \in C^{\psi}_{\beta,\infty}} \|\rho_\delta(f;\cdot)\|_C \tag{2}$$

as $\delta \to \infty.$

If a function $\varphi(\delta) = \varphi(\mathfrak{N}; \delta)$ such that

$$\mathcal{E}\left(\mathfrak{N}; B_\delta\right)_X = \varphi(\delta) + o(\varphi(\delta)) \quad \text{for} \quad \delta \to \infty$$

is found in explicit form, then, following Stepanets [5, p. 198], we say that the Kolmogorov–Nikol'skii problem is solved for a biharmonic Poisson integral on the class $\mathfrak{N}$ in the metric of the space $X.$

Note that the Kolmogorov–Nikol'skii problem was solved on the class W^1_∞ by Kaniev [6] and Pych [7]. Approximation properties of biharmonic Poisson integrals on other classes of functions were also studied by Falaleev [8], Amanov and Falaleev [9], Timan [1], Zhyhallo and Kharkevych [10–12], and Zastavnyi [13]. It should also be noted that, in [12], the Kolmogorov–Nikol'skii problem was solved for biharmonic Poisson integrals on the classes $C^{\psi}_{\beta,\infty}$ in the metric of the space C in the case of functions $\psi(\cdot)$ rapidly decreasing to zero. At the same time, of special interest are approximation properties of biharmonic Poisson integrals on classes of (ψ,β)-differentiable functions of low smoothness, i.e., functions $\psi(\cdot)$ such that

$$\int_1^\infty u\psi(u)du = \infty.$$

2. Some Estimates for Fourier-Type Integrals

Let $\Lambda = \{\lambda_\delta(k)\}$ be the set of functions of a natural argument depending on a parameter $\delta,$ which is defined on a set $E_\Lambda \subseteq \mathbb{R}$ that has at least one limit point $\delta_0,$ and let $\lambda_\delta(0) = 1 \ \ \forall \delta \in E_\Lambda.$ If $\delta \in \mathbb{N},$ then the numbers $\lambda_\delta(k)$ are elements of an infinite rectangular matrix $\Lambda = \{\lambda^{(n)}_k\},\ n,k = 0,1,\ldots,\ \lambda^{(n)}_0 = 1,\ n \in \mathbb{N} \cup \{0\},$ and under the additional condition $\lambda^{(n)}_k \equiv 0$ for $k > n,$ they are elements of an infinite triangular matrix. We assume that $\{\lambda_\delta(k)\}$ possesses the following property: For any function $f \in L_1$ and any fixed $\delta \in E_\Lambda,$ the series

$$\frac{a_0(f)}{2}\lambda_\delta(0) + \sum_{k=1}^{\infty} \lambda_\delta(k)\left(a_k(f)\cos kx + b_k(f)\sin kx\right), \quad \delta \in E_\Lambda,$$

converges to a summable function $U_\delta(f;x;\Lambda)$ in the metric of the space L_1. One says that, for a fixed $\delta \in E_\Lambda$, every set of functions of a natural argument Λ determines a linear operator $U_\delta(\Lambda)$ that acts from L_1 into L_1. In particular, for the biharmonic Poisson operator B_δ, we have

$$\lambda_\delta(k) = \left(1 + \frac{k}{2}\left(1 - e^{-2/\delta}\right)\right) e^{-k/\delta},$$

where $\delta > 0$ and $\delta_0 = \infty$ is a limit point of the set E_Λ.

Further, assume that the set Λ is determined by a summation function $\lambda_\delta(u)$, $0 \le u < \infty$, such that

$$\lambda_\delta(k) = \lambda\left(\frac{k}{\delta}\right) \quad \text{and} \quad \lambda_\delta(0) = 1 \quad \forall \delta \in E_\Lambda.$$

For the biharmonic Poisson integral, we set

$$\tau_\delta\left(\frac{k}{\delta}\right) = (1 - \lambda_\delta(k))\frac{\psi(k)}{\psi(\delta)}, \quad k = 0, 1, 2, \ldots,$$

so that

$$\tau(u) = \tau_\delta(u;\psi) = \begin{cases} (1 - [1 + \gamma u]\, e^{-u})\dfrac{\psi(1)}{\psi(\delta)}, & 0 \le u \le \dfrac{1}{\delta}, \\[2ex] (1 - [1 + \gamma u]\, e^{-u})\dfrac{\psi(\delta u)}{\psi(\delta)}, & u \ge \dfrac{1}{\delta}, \end{cases} \tag{3}$$

where

$$\gamma = \gamma(\delta) = \frac{\delta}{2}(1 - e^{-2/\delta})$$

and $\psi(u)$ is a function defined and continuous for $u \ge 1$.

Prior to passing to the investigation of the behavior of a quantity $\mathcal{E}\left(C^\psi_{\beta,\infty}; B_\delta\right)_C$ of the form (2), we prove the following statements:

Lemma 1. *If the Fourier transform*

$$\hat{\tau}(t) = \hat{\tau}_\delta(t) = \frac{1}{\pi}\int_0^\infty \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du \tag{4}$$

of a function $\tau(\cdot)$ of the form (3) is summable everywhere on the number axis, then

$$\mathcal{E}\left(C^\psi_{\beta,\infty}; B_\delta\right)_C = \psi(\delta)A(\tau) + O\left(\psi(\delta)\int\limits_{|t|\ge\delta\pi/2} |\hat{\tau}_\delta(t)|\, dt\right), \tag{5}$$

where

$$A(\tau) = \int_{-\infty}^{\infty} |\hat{\tau}_\delta(t)|\, dt. \tag{6}$$

Proof. Since, according to the conditions of Lemma 1, the Fourier transform $\hat{\tau}(\cdot)$ is summable everywhere on the number axis, by analogy with [5, p. 183] one can easily verify that, for any function $f \in C^{\psi}_{\beta,\infty}$, the following equality holds at any point $x \in \mathbb{R}$:

$$\rho_\delta(f;x) = f(x) - B_\delta(f;x) = \psi(\delta) \int_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right) \hat{\tau}_\delta(t)dt, \quad \delta > 0. \tag{7}$$

Using relation (2) and taking into account the integral representation (7) and the fact that the class $C^{\psi}_{\beta,\infty}$ is invariant under the shift of arguments (see [4, p. 109]), we obtain

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; B_\delta\right)_C = \sup_{f \in C^{\psi}_{\beta,\infty}} \left| \psi(\delta) \int_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(\frac{t}{\delta}\right) \hat{\tau}_\delta(t)dt \right|.$$

Hence,

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; B_\delta\right)_C \le \frac{\psi(\delta)}{\pi} \int_{-\infty}^{+\infty} \left| \int_0^{\infty} \tau(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt. \tag{8}$$

On the other hand, for any function $\varphi_0 \in L_1$ such that

$$\int_{-\pi}^{\pi} \varphi_0(t)dt = 0 \qquad \text{and} \qquad \operatorname*{ess\,sup}_t |\varphi_0(t)| \le 1,$$

there exists a function $f(x) = f(\varphi_0; x)$ in the class $C^{\psi}_{\beta,\infty}$ for which we have $f^{\psi}_{\beta}(x) = \varphi_0(x)$. Therefore, there exists a function $\hat{f}(t)$ in the class $C^{\psi}_{\beta,\infty}$ for which

$$\hat{f}^{\psi}_{\beta}(t) = \operatorname{sign} \int_0^{\infty} \tau(u) \cos\left(u\delta t + \frac{\beta\pi}{2}\right) du, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{9}$$

Furthermore, since

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; B_\delta\right)_C \ge \frac{\psi(\delta)}{\pi} \left| \int_{-\infty}^{+\infty} \hat{f}^{\psi}_{\beta}\left(\frac{t}{\delta}\right) \int_0^{\infty} \tau(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du dt \right|, \tag{10}$$

taking (9) into account we get

$$\frac{\psi(\delta)}{\pi}\left|\int_{-\infty}^{+\infty}\hat{f}_\beta^\psi\left(\frac{t}{\delta}\right)\int_0^\infty \tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\,dt\right|$$

$$\geq \delta\psi(\delta)\left|\int_{-\pi/2}^{\pi/2}\operatorname{sign}\hat{\tau}(t\delta)\hat{\tau}(t\delta)\,dt\right| - \psi(\delta)\int_{|t|\geq\delta\pi/2}|\hat{\tau}_\delta(t)|\,dt$$

$$= \psi(\delta)\int_{-\infty}^{+\infty}|\hat{\tau}_\delta(t)|\,dt + \gamma(\delta), \tag{11}$$

where $\gamma(\delta) \leq 0$ and

$$|\gamma(\delta)| = O\left(\psi(\delta)\int_{|t|\geq\delta\pi/2}|\hat{\tau}_\delta(t)|\,dt\right).$$

Combining relations (8), (10), and (11), we obtain equality (5).
Lemma 1 is proved.

Note that a similar result for triangular matrices Λ, $\lambda_k^{(n)} \equiv 0$, $k > n$, was established for the classes $W_{\beta,\infty}^r$ by Telyakovskiy [14] and for the classes $C_{\beta,\infty}^\psi$ by Rukasov in [15]. For infinite rectangular matrices $\Lambda = \{\lambda_k^{(n)}\}$, $n, k = 0, 1, \ldots,$ on the classes $W_{\beta,\infty}^r$, there is a known result obtained by Bausov [16].

In Lemma 1, one requires the summability of the transform $\hat{\tau}(t)$ of a function $\tau(\cdot)$ of the type (3) on the entire real axis, i.e., the convergence of the integral $A(\tau)$. According to Theorem 1 in [16], a necessary and sufficient condition for this requirement to be satisfied is the convergence of the following integrals:

$$\int_0^{1/2} u|d\tau'(u)|, \qquad \int_{1/2}^\infty |u-1||d\tau'(u)|, \tag{12}$$

$$\left|\sin\frac{\beta\pi}{2}\right|\int_0^\infty \frac{|\tau(u)|}{u}du, \qquad \int_0^1 \frac{|\tau(1-u)-\tau(1+u)|}{u}du. \tag{13}$$

Lemma 2. *If ψ belongs to the set $\mathfrak{M}_0'$ and the function $g(u) = u^2\psi(u)$ is convex either upward or downward on $[b,\infty)$, $b \geq 1$, then integrals (12) and (13), where $\tau(\cdot)$ is a function of the type (3), admit the following estimates as $\delta \to \infty$:*

$$\int_0^{1/2} u|d\tau'(u)| = O\left(1 + \frac{1}{\delta^2\psi(\delta)}\int_1^\delta \psi(u)du\right), \tag{14}$$

$$\int_{1/2}^{\infty} |u-1||d\tau'(u)| = O(1), \tag{15}$$

$$\int_{0}^{\infty} \frac{|\tau(u)|}{u} du = \frac{1}{2\delta^2\psi(\delta)} \int_{1}^{\delta} u\psi(u)du + \frac{1}{\psi(\delta)} \int_{\delta}^{\infty} \frac{\psi(u)}{u} du + O\left(1 + \frac{1}{\delta^2\psi(\delta)} \int_{1}^{\delta} \psi(u)du\right), \tag{16}$$

$$\int_{0}^{1} \frac{|\tau(1-u) - \tau(1+u)|}{u} du = O\left(1 + \frac{1}{\delta^2\psi(\delta)} \int_{1}^{\delta} \psi(u)du\right). \tag{17}$$

Proof. Let us estimate the first integral in (12) on the intervals $\left[0; \frac{1}{\delta}\right]$ and $\left[\frac{1}{\delta}; \frac{1}{2}\right]$ (for $\delta > 2b$). It follows from relation (3) for $u \in \left[0, \frac{1}{\delta}\right]$ that

$$\tau'(u) = e^{-u}\,(1-\gamma+\gamma u)\,\frac{\psi(1)}{\psi(\delta)}, \qquad \tau''(u) = e^{-u}\,(-1+2\gamma-\gamma u)\,\frac{\psi(1)}{\psi(\delta)}.$$

Note that

$$-1+2\gamma-\gamma u > 0, \quad u \in \left[0, \frac{1}{\delta}\right],$$

for sufficiently large δ, and

$$1-\gamma+\gamma u > 0$$

for $0 < \gamma < 1$ and $u > 0$. Taking this into account, we conclude that the function $\tau(u)$ is convex downward for $u \in \left[0; \frac{1}{\delta}\right]$. Therefore, using the inequalities

$$\gamma < 1, \quad 1-\gamma < \frac{1}{\delta}, \tag{18}$$

$$1 - e^{-u} - \gamma u e^{-u} < \frac{u}{\delta} + u^2, \quad u \ge 0, \tag{19}$$

one can easily verify that

$$\int_{0}^{1/\delta} u|d\tau'(u)| \le \frac{K}{\delta^2\psi(\delta)}. \tag{20}$$

We set

$$\tau(u) = \tau_1(u) + \tau_2(u) + \tau_3(u), \quad u \ge \frac{1}{\delta},$$

where

$$\tau_1(u) := \left(1 - e^{-u} - \gamma u e^{-u} - \frac{u}{\delta} - \frac{u^2}{2}\right) \frac{\psi(\delta u)}{\psi(\delta)}, \tag{21}$$

$$\tau_2(u) := \frac{u}{\delta} \frac{\psi(\delta u)}{\psi(\delta)}, \tag{22}$$

$$\tau_3(u) := \frac{u^2}{2} \frac{\psi(\delta u)}{\psi(\delta)}. \tag{23}$$

Then

$$\int\limits_{1/\delta}^{1/2} u|d\tau'(u)| \leq \int\limits_{1/\delta}^{1/2} u|d\tau_1'(u)| + \int\limits_{1/\delta}^{1/2} u|d\tau_2'(u)| + \int\limits_{1/\delta}^{1/2} u|d\tau_3'(u)|. \tag{24}$$

Let us estimate the first integral on the right-hand side of (24). To this end, we first consider the following function:

$$\overline{\mu}(u) = 1 - e^{-u} - \gamma u e^{-u} - \frac{u^2}{2} - \frac{u}{\delta}. \tag{25}$$

It follows from the relations

$$\overline{\mu}'(u) = e^{-u} - \gamma e^{-u} + \gamma u e^{-u} - u - 1/\delta,$$

$$\overline{\mu}''(u) = -e^{-u} + 2\gamma e^{-u} - \gamma u e^{-u} - 1,$$

$$\overline{\mu}(0) = 0,\ \overline{\mu}'(0) = 1 - \gamma - 1/\delta < 0,$$

$$-1 + 2\gamma - \gamma u < e^u, \quad u \in [0, \infty),$$

that, for $u \geq 0$, we have

$$\overline{\mu}(u) \leq 0, \qquad \overline{\mu}'(u) < 0, \qquad \overline{\mu}''(u) < 0. \tag{26}$$

Taking into account relation (26) and the fact that

$$e^{-u} \leq 1 - u + \frac{u^2}{2}, \quad e^{-u} \geq 1 - u,$$

we get

$$|\overline{\mu}(u)| = \frac{u^2}{2} + \frac{u}{\delta} - 1 + e^{-u} + \gamma u e^{-u}$$

$$\leq \frac{u^2}{2} + \frac{u}{\delta} - u + \frac{u^2}{2} + \gamma u - \gamma u^2 + \gamma \frac{u^3}{2}$$

$$= (-1 + \gamma + 1/\delta)\, u + (1-\gamma)\, u^2 + \gamma \frac{u^3}{2},$$

$$\left|\overline{\mu}'(u)\right| = u + 1/\delta - e^{-u} + \gamma e^{-u} - \gamma u e^{-u}$$

$$\leq u + 1/\delta - 1 + u + \gamma\left(1 - u + \frac{u^2}{2}\right) - \gamma u + \gamma u^2$$

$$= (-1 + \gamma + 1/\delta) + 2(1-\gamma)u + \frac{3}{2}\gamma u^2,$$

$$\left|\overline{\mu}''(u)\right| = e^{-u} - 2\gamma e^{-u} + \gamma u e^{-u} + 1$$

$$\leq 1 - 2\gamma + 2\gamma u + \gamma u + 1 = (2-2\gamma) + 3\gamma u.$$

This, by virtue of relation (18) and the inequality

$$-1 + \gamma + \frac{1}{\delta} < \frac{2}{3\delta^2},$$

yields

$$|\overline{\mu}(u)| < \frac{2}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{u^3}{2},$$

$$\left|\overline{\mu}'(u)\right| < \frac{2}{3\delta^2} + \frac{2}{\delta}u + \frac{3}{2}u^2, \tag{27}$$

$$\left|\overline{\mu}''(u)\right| < \frac{2}{\delta} + 3u.$$

According to (21) and (25), the following relation holds for $u \geq \frac{1}{\delta}$:

$$|d\tau_1'(u)| \leq \left\{|\overline{\mu}(u)|\,\frac{\delta^2\psi''(\delta u)}{\psi(\delta)} + 2\left|\overline{\mu}'(u)\right|\frac{\delta|\psi'(\delta u)|}{\psi(\delta)} + \left|\overline{\mu}''(u)\right|\frac{\psi(\delta u)}{\psi(\delta)}\right\} du. \tag{28}$$

Taking this and (27) into account, we obtain

$$\int\limits_{1/\delta}^{1/2} u|d\tau_1'(u)| \leq \frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} \left(\frac{2}{3\delta^2}u^2 + \frac{1}{\delta}u^3 + \frac{1}{2}u^4\right)\delta^2\psi''(\delta u)du$$

$$+ \frac{2}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} \left(\frac{2}{3\delta^2}u + \frac{2}{\delta}u^2 + \frac{3}{2}u^3\right)\delta\left|\psi'(\delta u)\right|du + \frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} \left(\frac{2}{\delta}u + 3u^2\right)\psi(\delta u)du.$$

Integrating the first integral on the right-hand side of the last inequality by parts, we get

$$\int\limits_{1/\delta}^{1/2} u|d\tau_1'(u)| \leq \frac{1}{\psi(\delta)} \left(\frac{2}{3\delta^2}u^2 + \frac{1}{\delta}u^3 + \frac{1}{2}u^4\right)\delta\psi'(\delta u)\Bigg|_{1/\delta}^{1/2}$$

$$+ \frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} \left(\frac{8}{3\delta^2}u + \frac{7}{\delta}u^2 + 5u^3\right)\delta\left|\psi'(\delta u)\right|du + \frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} \left(\frac{2}{\delta}u + 3u^2\right)\psi(\delta u)du. \quad (29)$$

We also need the following statements:

Theorem 1′ [5, p. 161]. *A function $\psi \in \mathfrak{M}$ belongs to the set $\mathfrak{M}_0$ if and only if the quantity*

$$\alpha(t) = \frac{\psi(t)}{t\,|\psi'(t)|}, \qquad \psi'(t) = \psi'(t+0), \quad (30)$$

satisfies the condition $\alpha(t) \geq K > 0 \ \ \forall t \geq 1$.

Theorem 2′ [5, p. 175]. *In order that a function $\psi \in \mathfrak{M}$ belong to the set $\mathfrak{M}_0$, it is necessary and sufficient that, for any fixed number $c > 1$, there exist a constant K such that*

$$\frac{\psi(t)}{\psi(ct)} \leq K \quad \textit{for all} \;\; t \geq 1.$$

In what follows, K and K_i denote certain constants (generally speaking, different).

Using Theorem 1′, for any function $\psi \in \mathfrak{M}_0$ we get

$$\frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} \left(\frac{8}{3\delta^2}u + \frac{7}{\delta}u^2 + 5u^3\right)\delta\left|\psi'(\delta u)\right|du \leq \frac{K}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} \left(\frac{8}{3\delta^2} + \frac{7}{\delta}u + 5u^2\right)\psi(\delta u)du.$$

Using relation (29), the last estimate, and Theorem 1′, we obtain

$$\int_{1/\delta}^{1/2} u|d\tau_1'(u)| \leq K_1 + \frac{K_2}{\delta^3\psi(\delta)} + \frac{K_3}{\delta^2\psi(\delta)} \int_{1/\delta}^{1/2} \psi(\delta u)du$$

$$+ \frac{K_4}{\delta\psi(\delta)} \int_{1/\delta}^{1/2} u\psi(\delta u)du + \frac{K_5}{\psi(\delta)} \int_{1/\delta}^{1/2} u^2\psi(\delta u)du. \tag{31}$$

Consider the integral

$$\frac{1}{\psi(\delta)} \int_{1/\delta}^{1/2} u^2\psi(\delta u)du = \frac{1}{\psi(\delta)} \left(\int_{1/\delta}^{b/\delta} + \int_{b/\delta}^{1/2} \right) u^2\psi(\delta u)du, \quad \delta > 2b.$$

Since the function $g(u) = u^2\psi(u)$ is bounded on $[1, b]$ and convex for $u \geq b \geq 1$, we have

$$\frac{1}{\psi(\delta)} \int_{1/\delta}^{1/2} u^2\psi(\delta u)du = \frac{1}{\delta^3\psi(\delta)} \int_{1}^{\delta/2} u^2\psi(u)du$$

$$= \frac{1}{\delta^3\psi(\delta)} \left(\int_{1}^{b} + \int_{b}^{\delta/2} \right) u^2\psi(u)du$$

$$\leq \frac{1}{\delta^3\psi(\delta)} \left(\int_{1}^{b} + \int_{b}^{\delta} \right) u^2\psi(u)du = O\left(1 + \frac{1}{\delta^2\psi(\delta)}\right). \tag{32}$$

Then, taking into account the inequality

$$\frac{1}{\delta^2} \int_{1/\delta}^{1/2} \psi(\delta u)du \leq \frac{1}{\delta} \int_{1/\delta}^{1/2} u\psi(\delta u)du \leq \int_{1/\delta}^{1/2} u^2\psi(\delta u)du,$$

and relations (31) and (32), we obtain

$$\int_{1/\delta}^{1/2} u|d\tau_1'(u)| = O\left(1 + \frac{1}{\delta^2\psi(\delta)}\right). \tag{33}$$

For $u \geq 1/\delta$, equality (22) yields

$$\psi(\delta) d\tau_2'(u) = u\delta\psi''(\delta u) + 2\psi'(\delta u), \tag{34}$$

whence

$$\int_{1/\delta}^{1/2} u|d\tau_2'(u)| \leq \frac{1}{\psi(\delta)} \int_{1/\delta}^{1/2} u^2\delta\psi''(\delta u)du + \frac{2}{\psi(\delta)} \int_{1/\delta}^{1/2} u\left|\psi'(\delta u)\right| du.$$

Integrating the first integral in the last inequality by parts and taking Theorems $1'$ and $2'$ into account, we obtain

$$\int_{1/\delta}^{1/2} u|d\tau_2'(u)| \leq \frac{1}{\psi(\delta)} u^2\psi'(\delta u)\Big|_{1/\delta}^{1/2} + \frac{4}{\psi(\delta)} \int_{1/\delta}^{1/2} u\left|\psi'(\delta u)\right| du$$

$$\leq K_1 + \frac{K_2}{\delta^2\psi(\delta)} + \frac{K_3}{\delta\psi(\delta)} \int_{1/\delta}^{1/2} \psi(\delta u)du = O\left(1 + \frac{1}{\delta^2\psi(\delta)} \int_1^{\delta} \psi(u)du\right). \tag{35}$$

Let us estimate the third term on the right-hand side of inequality (24) on each of the segments $\left[\frac{1}{\delta}, \frac{b}{\delta}\right]$ and $\left[\frac{b}{\delta}, \frac{1}{2}\right]$, $\delta > 2b$. Using (23), we determine $\frac{d\tau_3'(u)}{du}$. Taking into account that the function $\psi(\delta u)$ is decreasing and convex downward on the segment $\left[\frac{1}{\delta}, \frac{b}{\delta}\right]$, we obtain

$$\int_{1/\delta}^{b/\delta} u|d\tau_3'(u)| \leq \frac{1}{\psi(\delta)} \left(\int_{1/\delta}^{b/\delta} u\psi(\delta u)du + 2\delta \int_{1/\delta}^{b/\delta} u^2|\psi'(\delta u)|du + \delta^2 \int_{1/\delta}^{b/\delta} u^3\psi''(\delta u)du \right). \tag{36}$$

Since $\psi(\delta u) \leq \psi(1)$ for $u \in \left[\frac{1}{\delta}, \frac{b}{\delta}\right]$, we have

$$\frac{1}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u\psi(\delta u)du \leq \frac{\psi(1)}{\psi(\delta)} \int_{1/\delta}^{b/\delta} udu = \frac{K}{\delta^2\psi(\delta)}. \tag{37}$$

Taking into account Theorem $1'$ and relation (37), we obtain

$$\frac{\delta}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u^2|\psi'(\delta u)|du \leq \frac{K_1}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u\psi(\delta u)du \leq \frac{K_2}{\delta^2\psi(\delta)}. \tag{38}$$

Integrating the third integral on the right-hand side of (36) by parts with regard for (37) and (38), we get

$$\frac{\delta^2}{\psi(\delta)}\int\limits_{1/\delta}^{b/\delta} u^3\psi''(\delta u)du \le \frac{K_2}{\delta^2\psi(\delta)}. \tag{39}$$

Combining relations (36)–(39), we obtain

$$\int\limits_{1/\delta}^{b/\delta} u|d\tau_3'(u)| = O\left(\frac{1}{\delta^2\psi(\delta)}\right). \tag{40}$$

It follows from relation (23) and the convexity of the function $g(u)$ for $u \ge b,\ b \ge 1,$ that

$$\int\limits_{b/\delta}^{1/2} u|d\tau_3'(u)| = \left|\int\limits_{b/\delta}^{1/2} u\,d\tau_3'(u)\right| = \left|\left(u\tau_3'(u) - \tau_3(u)\right)\Big|_{b/\delta}^{1/2}\right| = O\left(1 + \frac{1}{\delta^2\psi(\delta)}\right). \tag{41}$$

Therefore, according to relations (20), (24), (33), and (35) and equalities (40) and (41) for sufficiently large δ, equality (14) is true.

Let us estimate the second integral in (12). According to (3), for $u \in [1/\delta;\infty)$ we have

$$\psi(\delta)d\tau'(u) = \Big\{\left(1 - [1+\gamma u]\,e^{-u}\right)\delta^2\psi''(\delta u) + 2\delta\left(e^{-u} - \gamma e^{-u} + \gamma u e^{-u}\right)\psi'(\delta u)$$

$$+ \left(-e^{-u} + 2\gamma e^{-u} - \gamma u e^{-u}\right)\psi(\delta u)\Big\}du. \tag{42}$$

Therefore,

$$\int\limits_{1/2}^{\infty} |u-1||d\tau'(u)| \le \int\limits_{1/2}^{\infty} u|d\tau'(u)|$$

$$\le \frac{1}{\psi(\delta)}\int\limits_{1/2}^{\infty} u\left(1 - [1+\gamma u]\,e^{-u}\right)\delta^2\psi''(\delta u)du$$

$$+ \frac{2\delta}{\psi(\delta)}\int\limits_{1/2}^{\infty} u e^{-u}(1-\gamma+\gamma u)\left|\psi'(\delta u)\right|du$$

$$+ \frac{1}{\psi(\delta)}\int\limits_{1/2}^{\infty} u e^{-u}\left|-1+2\gamma-\gamma u\right|\psi(\delta u)du.$$

Further, taking into account that

$$1-[1+\gamma u]\,e^{-u}\leq 1 \quad \text{and} \quad ue^{-u}\,(1-\gamma+\gamma u)\leq K \quad \text{for} \quad u\geq 0$$

and

$$\psi(\delta u)\leq \psi\left(\frac{\delta}{2}\right) \quad \text{for} \quad u\in\left[\frac{1}{2};+\infty\right),$$

one can verify that relation (15) holds as $\delta\to\infty$.

Let us estimate the first integral in (13) on the segments $\left[0;\frac{1}{\delta}\right]$, $\left[\frac{1}{\delta};1\right]$, and $[1,\infty)$. By virtue of (3) and (19), we have

$$\int\limits_0^{1/\delta}\frac{\tau(u)}{u}du\leq\frac{\psi(1)}{\psi(\delta)}\int\limits_0^{1/\delta}\left(\frac{u}{\delta}+u^2\right)\frac{du}{u}\leq\frac{K}{\delta^2\psi(\delta)}. \tag{43}$$

Taking into account (3), (25), and (27), we get

$$\left|\int\limits_{1/\delta}^{1}\frac{\tau(u)}{u}du-\frac{1}{2\psi(\delta)}\int\limits_{1/\delta}^{1}u\psi(\delta u)du-\frac{1}{\delta\psi(\delta)}\int\limits_{1/\delta}^{1}\psi(\delta u)du\right|\leq\frac{1}{\psi(\delta)}\int\limits_{1/\delta}^{1}\frac{|\overline{\mu}(u)|}{u}\psi(\delta u)du\leq K_1+\frac{K_2}{\delta^2\psi(\delta)}.$$

Hence,

$$\int\limits_{1/\delta}^{1}\frac{\tau(u)}{u}du=\frac{1}{2\delta^2\psi(\delta)}\int\limits_1^{\delta}u\psi(u)du+O\left(1+\frac{1}{\delta^2\psi(\delta)}\int\limits_1^{\delta}\psi(u)du\right). \tag{44}$$

Using equality (3) once again and taking into account that $\psi(\delta u)\leq\psi(\delta)$ for $u\geq 1$, we get

$$\left|\int\limits_1^{\infty}\frac{\tau(u)}{u}du-\frac{1}{\psi(\delta)}\int\limits_{\delta}^{\infty}\frac{\psi(u)}{u}du\right|\leq\frac{1}{\psi(\delta)}\int\limits_1^{\infty}\frac{\psi(\delta u)}{u}\left(e^{-u}+\gamma ue^{-u}\right)du\leq K. \tag{45}$$

Combining (43)–(45), we obtain relation (16).

Let us estimate the second integral in (13). We set

$$\lambda_\delta(u)=[1+u\gamma(\delta)]\,e^{-u}=\left[1+\frac{\delta u}{2}\left(1-e^{-2/\delta}\right)\right]e^{-u}. \tag{46}$$

Then the function $\tau(\cdot)$ defined by (3) takes the form

$$\tau(u)=\begin{cases}(1-\lambda_\delta(u))\,\dfrac{\psi(1)}{\psi(\delta)}, & 0\leq u\leq\dfrac{1}{\delta},\\[2ex] (1-\lambda_\delta(u))\,\dfrac{\psi(\delta u)}{\psi(\delta)}, & u\geq\dfrac{1}{\delta}.\end{cases} \tag{47}$$

Using (47), we get

$$\tau(1-u)=\begin{cases}(1-\lambda_\delta(1-u))\,\dfrac{\psi(1)}{\psi(\delta)}, & 1-\dfrac{1}{\delta}\le u\le 1,\\[2ex] (1-\lambda_\delta(1-u))\,\dfrac{\psi(\delta(1-u))}{\psi(\delta)}, & u\le 1-\dfrac{1}{\delta},\end{cases} \tag{48}$$

$$\tau(1+u)=\begin{cases}(1-\lambda_\delta(1+u))\,\dfrac{\psi(1)}{\psi(\delta)}, & -1\le u\le \dfrac{1}{\delta}-1,\\[2ex] (1-\lambda_\delta(1+u))\,\dfrac{\psi(\delta(1+u))}{\psi(\delta)}, & u\ge \dfrac{1}{\delta}-1.\end{cases} \tag{49}$$

We represent the second integral in (13) as a sum of two integrals as follows:

$$\int\limits_0^1 \frac{|\tau(1-u)-\tau(1+u)|}{u}du=\int\limits_0^{1-1/\delta} \frac{|\tau(1-u)-\tau(1+u)|}{u}du+\int\limits_{1-1/\delta}^{1} \frac{|\tau(1-u)-\tau(1+u)|}{u}du. \tag{50}$$

Adding and subtracting the value $\lambda_\delta(1-u)-\lambda_\delta(1+u)$ under the modulus sign in the integrand of the first term on the right-hand side of (50), we obtain

$$\int\limits_0^{1-1/\delta} \frac{|\tau(1-u)-\tau(1+u)|}{u}du\le\int\limits_0^{1-1/\delta} \frac{|\lambda_\delta(1-u)-\lambda_\delta(1+u)|}{u}du$$
$$+\int\limits_0^{1-1/\delta} \frac{|\tau(1-u)-\tau(1+u)+\lambda_\delta(1-u)-\lambda_\delta(1+u)|}{u}du. \tag{51}$$

One can easily verify that the first integral on the right-hand side of inequality (51), where $\lambda_\delta(u)$ is a function of the type (46), admits the following estimate:

$$\int\limits_0^{1-1/\delta} \left|(1+\gamma(1-u))\,e^{-1+u}-(1+\gamma(1+u))\,e^{-1-u}\right|\frac{du}{u}=O(1). \tag{52}$$

Furthermore, by virtue of relations (48) and (49), for $u\in\left[0,1-\dfrac{1}{\delta}\right]$ we have

$$\lambda_\delta(1-u)=1-\frac{\psi(\delta)}{\psi(\delta(1-u))}\tau(1-u),$$

$$\lambda_\delta(1+u)=1-\frac{\psi(\delta)}{\psi(\delta(1+u))}\tau(1+u).$$

Then

$$\int_0^{1-1/\delta} |\tau(1-u) - \tau(1+u) + (\lambda_\delta(1-u) - \lambda_\delta(1+u))| \frac{du}{u}$$

$$\le \int_0^{1-1/\delta} |\tau(1-u)| \left|1 - \frac{\psi(\delta)}{\psi(\delta(1-u))}\right| \frac{du}{u} + \int_0^{1-1/\delta} |\tau(1+u)| \left|1 - \frac{\psi(\delta)}{\psi(\delta(1+u))}\right| \frac{du}{u}. \tag{53}$$

For the estimation of the integrals on the right-hand side of (53), we use statements established by Bausov in [16].

Definition 1′ [16]. *Suppose that a function* $\tau(u)$ *is defined on* $[0, \infty)$, *absolutely continuous, and such that* $\tau(\infty) = 0$. *One says that* $\tau(u)$ *belongs to a set* $\mathcal{E}_1$ *if the definition of the derivative* $\tau'(u)$ *can be extended to the points where it does not exist so that the following integrals exist:*

$$\int_0^{1/2} u|d\tau'(u)| \qquad \text{and} \qquad \int_{1/2}^{\infty} |u-1||d\tau'(u)|.$$

Let

$$H(\tau) = |\tau(0)| + |\tau(1)| + \int_0^{1/2} u\left|d\tau'(u)\right| + \int_{1/2}^{\infty} |u-1|\left|d\tau'(u)\right|. \tag{54}$$

Lemma 1′ [16]. *If* $\tau(u)$ *belongs to* $\mathcal{E}_1$, *then* $|\tau(u)| \le H(\tau)$.

Since the function $\tau(\cdot)$ defined by (3) belongs to the set $\mathcal{E}_1$, we can use Lemma 1′, according to which

$$\int_0^{1-1/\delta} |\tau(1-u)| \left|1 - \frac{\psi(\delta)}{\psi(\delta(1-u))}\right| \frac{du}{u} + \int_0^{1-1/\delta} |\tau(1+u)| \left|1 - \frac{\psi(\delta)}{\psi(\delta(1+u))}\right| \frac{du}{u}$$

$$= H(\tau) O\left(\int_0^{1-1/\delta} \frac{|\psi(\delta(1-u)) - \psi(\delta)|}{u\psi(\delta(1-u))} du + \int_0^{1-1/\delta} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u\psi(\delta(1+u))} du \right). \tag{55}$$

Following [17], we can prove that, for functions $\psi \in \mathfrak{M}_0$, both integrals on the right-hand side of (55) are of order $O(1)$ for $\delta \to \infty$, i.e., they are uniformly bounded with respect to δ. Thus, it follows from (53) and (55) that

$$\int_0^{1-1/\delta} \left|\tau(1-u) - \tau(1+u) + (\lambda_\delta(1-u) - \lambda_\delta(1+u))\right| \frac{du}{u} = H(\tau) O(1). \tag{56}$$

Moreover, for a quantity $H(\tau)$ of the type (54), the following estimate holds by virtue of (3), (14), and (15):

$$H(\tau) = O\left(1 + \frac{1}{\delta^2 \psi(\delta)} \int_1^\delta \psi(u)du\right), \quad \delta \to \infty. \tag{57}$$

Comparing (51) with (52), (56), and (57), we conclude that

$$\int_0^{1-1/\delta} \frac{|\tau(1-u) - \tau(1+u)|}{u} du = O\left(1 + \frac{1}{\delta^2 \psi(\delta)} \int_1^\delta \psi(u)du\right). \tag{58}$$

Let us estimate the second term on the right-hand side of (50). We have

$$\int_{1-1/\delta}^{1} \frac{|\tau(1-u) - \tau(1+u)|}{u} du = \int_{1-1/\delta}^{1} \frac{|\lambda_\delta(1-u) - \lambda_\delta(1+u)|}{u} du$$

$$+ O\left(\int_{1-1/\delta}^{1} |\tau(1-u) - \tau(1+u) + \lambda_\delta(1-u) - \lambda_\delta(1+u)| \frac{du}{u}\right). \tag{59}$$

For $u \in \left[1 - \frac{1}{\delta}, 1\right]$, relations (48) and (49) yield

$$\lambda_\delta(1-u) = 1 - \frac{\psi(\delta)}{\psi(1)}\tau(1-u), \qquad \lambda_\delta(1+u) = 1 - \frac{\psi(\delta)}{\psi(\delta(1+u))}\tau(1+u).$$

Hence, according to Lemma 1′, we get

$$\int_{1-1/\delta}^{1} |\tau(1-u) - \tau(1+u) + \lambda_\delta(1-u) - \lambda_\delta(1+u)| \frac{du}{u}$$

$$= \int_{1-1/\delta}^{1} \left|\tau(1-u)\left(1 - \frac{\psi(\delta)}{\psi(1)}\right) - \tau(1+u)\left(1 - \frac{\psi(\delta)}{\psi(\delta(1+u))}\right)\right| \frac{du}{u}$$

$$= H(\tau) O\left(\int_{1-1/\delta}^{1} \frac{|\psi(1) - \psi(\delta)|}{u\psi(1)} du + \int_{1-1/\delta}^{1} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u\psi(\delta(1+u))} du\right). \tag{60}$$

For $\delta \to \infty$, we have

$$\int_{1-1/\delta}^{1} \frac{|\psi(1) - \psi(\delta)|}{u\psi(1)} du = O(1),$$

$$\int_{1-1/\delta}^{1} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u\psi(\delta(1+u))} du = O(1),$$

where $O(1)$ is a quantity uniformly bounded with respect to δ. Therefore, taking into account that

$$\int_{1-1/\delta}^{1} \frac{|\lambda_\delta(1-u) - \lambda_\delta(1+u)|}{u} du = \int_{1-1/\delta}^{1} \left| e^{-1+u} - e^{-1-u} + \gamma(1-u)e^{-1+u} - \gamma(1+u)e^{-1-u} \right| \frac{du}{u} = O(1)$$

and using relations (57), (59), and (60), we get

$$\int_{1-1/\delta}^{1} \frac{|\tau(1-u) - \tau(1+u)|}{u} du = O\left(1 + \frac{1}{\delta^2\psi(\delta)} \int_{1}^{\delta} \psi(u)du\right), \quad \delta \to \infty. \tag{61}$$

By virtue of (58) and (61), equality (50) yields relation (17).

Theorem 2 is proved.

Thus, by virtue of Lemma 2 and Theorem 1 [16], we can conclude that an integral $A(\tau)$ of the type (6) is convergent.

3. Asymptotic Equalities for Upper Bounds of Deviations of Biharmonic Poisson Integrals from Functions of the Classes $C^{\psi}_{\beta,\infty}$

The statement below is the main result of the present paper.

Theorem 1. *Suppose that $\psi \in \mathfrak{M}_0'$ and the function $g(u) = u^2\psi(u)$ is convex either upward or downward on $[b, \infty)$, $b \geq 1$. Then the following equality holds as $\delta \to \infty$:*

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; B_\delta\right)_C = \psi(\delta)A(\tau) + O\left(\frac{1}{\delta^2} + \frac{1}{\delta^3}\int_{1}^{\delta} u\psi(u)du\right), \tag{62}$$

where $A(\tau)$ is defined by (6) and admits the following estimate:

$$A(\tau) = \frac{1}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\left(\frac{1}{\delta^2\psi(\delta)}\int_{1}^{\delta} u\psi(u)du + \frac{2}{\psi(\delta)}\int_{\delta}^{\infty}\frac{\psi(u)}{u}du\right) + O\left(1 + \frac{1}{\delta^2\psi(\delta)}\int_{1}^{\delta}\psi(u)du\right). \tag{63}$$

Proof. It follows from Lemma 1 that equality (5) is true. Moreover, in view of relations (14)–(17) for $A(\tau)$, inequalities (2.14) and (2.15) from [16, p. 25] yield estimate (63).

Let us estimate the remainder on the right-hand side of (5). For this purpose, we represent the transform $\hat{\tau}(t)$ as follows:

$$\hat{\tau}(t) = \frac{1}{\pi}\left(\int_0^{1/\delta} + \int_{1/\delta}^{\infty}\right)\tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du. \tag{64}$$

Integrating both integrals in (64) twice by parts and taking into account that $\tau(0) = 0$ and

$$\lim_{u\to\infty}\tau(u) = \lim_{u\to\infty}\tau'(u) = 0,$$

we obtain

$$\int_0^{1/\delta}\tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du = \frac{1}{t}\tau\left(\frac{1}{\delta}\right)\sin\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) + \frac{1}{t^2}\tau'\left(\frac{1}{\delta}\right)\cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right)$$

$$- \frac{1}{t^2}\tau'(0)\cos\frac{\beta\pi}{2} - \frac{1}{t^2}\int_0^{1/\delta}\tau''(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du, \tag{65}$$

$$\int_{1/\delta}^{\infty}\tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du = -\frac{1}{t}\tau\left(\frac{1}{\delta}\right)\sin\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2}\tau'\left(\frac{1}{\delta}\right)\cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right)$$

$$- \frac{1}{t^2}\int_{1/\delta}^{\infty}\tau''(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du. \tag{66}$$

Combining (65) and (66), we get

$$\int_0^{\infty}\tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du = -\frac{1}{t^2}\tau'(0)\cos\frac{\beta\pi}{2} - \frac{1}{t^2}\int_0^{1/\delta}\tau''(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du$$

$$- \frac{1}{t^2}\int_{1/\delta}^{\infty}\tau''(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du.$$

Since

$$\tau'(0) = (1-\gamma)\frac{\psi(1)}{\psi(\delta)} < \frac{\psi(1)}{\delta\psi(\delta)},$$

we have

$$\left|\int_0^\infty \tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right| \le \frac{K}{t^2\delta\psi(\delta)}+\frac{1}{t^2}\left(\int_0^{1/\delta}+\int_{1/\delta}^{1}+\int_1^\infty\right)\left|\tau''(u)\right|du. \tag{67}$$

Let us estimate integrals on the right-hand side of (67). Taking into account that the function $\tau(u)$ is convex downward for $u\in\left[0;\frac{1}{\delta}\right]$ and using inequalities (18), we obtain

$$\int_0^{1/\delta}|\tau''(u)|du = O\left(\frac{1}{\delta\psi(\delta)}\right). \tag{68}$$

It follows from (3) and (21)–(23) that

$$\int_{1/\delta}^{1}|\tau''(u)|du \le \int_{1/\delta}^{1}|\tau_1''(u)|du+\int_{1/\delta}^{1}|\tau_2''(u)|du+\int_{1/\delta}^{1}|\tau_3''(u)|du. \tag{69}$$

Inequalities (27) and (28) yield

$$\int_{1/\delta}^{1}|\tau_1''(u)|du \le \frac{1}{\psi(\delta)}\int_{1/\delta}^{1}\left(\frac{2}{3\delta^2}u+\frac{1}{\delta}u^2+\frac{1}{2}u^3\right)\delta^2\psi''(\delta u)du$$

$$+\frac{2}{\psi(\delta)}\int_{1/\delta}^{1}\left(\frac{2}{3\delta^2}+\frac{2}{\delta}u+\frac{3}{2}u^2\right)\delta\left|\psi'(\delta u)\right|du+\frac{1}{\psi(\delta)}\int_{1/\delta}^{1}\left(\frac{2}{\delta}+3u\right)\psi(\delta u)du.$$

Integrating the first integral on the right-hand side of the last inequality by parts and using Theorem $1'$, we get

$$\int_{1/\delta}^{1}|\tau_1''(u)|du \le \frac{1}{\psi(\delta)}\left(\frac{2}{3\delta^2}u+\frac{1}{\delta}u^2+\frac{1}{2}u^3\right)\delta\psi'(\delta u)\Bigg|_{1/\delta}^{1}$$

$$+\frac{3}{\psi(\delta)}\int_{1/\delta}^{1}\left(\frac{2}{3\delta^2}+\frac{2}{\delta}u+3u^2\right)\delta\left|\psi'(\delta u)\right|du$$

$$+\frac{1}{\psi(\delta)}\int_{1/\delta}^{1}\left(\frac{2}{\delta}+3u\right)\psi(\delta u)du \le K_1+\frac{K_2}{\delta^2\psi(\delta)}$$

$$-\frac{K_3}{\delta\psi(\delta)}\int\limits_{1/\delta}^{1}\psi'(\delta u)du+\frac{K_4}{\delta\psi(\delta)}\int\limits_{1/\delta}^{1}\psi(\delta u)du$$

$$+\frac{K_5}{\psi(\delta)}\int\limits_{1/\delta}^{1}u\psi(\delta u)du=O\left(\frac{1}{\delta^2\psi(\delta)}\int\limits_{1}^{\delta}u\psi(u)du\right). \tag{70}$$

Using relation (34), we obtain

$$\int\limits_{1/\delta}^{1}|\tau_2''(u)|du\le\frac{1}{\psi(\delta)}\int\limits_{1/\delta}^{1}u\delta\psi''(\delta u)du+\frac{2}{\psi(\delta)}\int\limits_{1/\delta}^{1}\left|\psi'(\delta u)\right|du$$

$$=\frac{1}{\psi(\delta)}\,u\psi'(\delta u)\Big|_{1/\delta}^{1}-\frac{3}{\psi(\delta)}\int\limits_{1/\delta}^{1}\psi'(\delta u)du=O\left(\frac{1}{\delta\psi(\delta)}\right). \tag{71}$$

Let us estimate the third integral in (69). For this purpose, we represent it as follows:

$$\int\limits_{1/\delta}^{1}|\tau_3''(u)|du=\left(\int\limits_{1/\delta}^{b/\delta}+\int\limits_{b/\delta}^{1}\right)|\tau_3''(u)|du,\quad \delta>b.$$

Then, by analogy with the proof of (36)–(40), one can easily verify that

$$\int\limits_{1/\delta}^{b/\delta}|\tau_3''(u)|du=O\left(\frac{1}{\delta\psi(\delta)}\right),\quad \delta\to\infty. \tag{72}$$

Taking into account relation (23) and the fact that the function $g(u)$ is convex on $[b,\infty)$, $b\ge 1$, we obtain

$$\int\limits_{b/\delta}^{1}|\tau_3''(u)|du=\left|\int\limits_{b/\delta}^{1}\tau_3''(u)du\right|=O\left(1+\frac{1}{\delta\psi(\delta)}\right). \tag{73}$$

Relations (69)–(73) yield

$$\int\limits_{1/\delta}^{1}|\tau''(u)|du=O\left(\frac{1}{\delta\psi(\delta)}+\frac{1}{\delta^2\psi(\delta)}\int\limits_{1}^{\delta}u\psi(u)du\right). \tag{74}$$

Using relations (42), we obtain an estimate for the third integral on the right-hand side of (67). Thus,

$$\int_1^\infty |\tau''(u)|du \le \frac{1}{\psi(\delta)}\int_1^\infty \left(1-[1+\gamma u]\,e^{-u}\right)\delta^2\psi''(\delta u)du$$

$$+\frac{2\delta}{\psi(\delta)}\int_1^\infty e^{-u}\,(1-\gamma+\gamma u)\left|\psi'(\delta u)\right| du$$

$$+\frac{1}{\psi(\delta)}\int_1^\infty e^{-u}\,|-1+2\gamma-\gamma u|\,\psi(\delta u)du.$$

Then, taking into account that

$$1-[1+\gamma u]\,e^{-u}\le u,\quad e^{-u}\,(1-\gamma+\gamma u)\le K,\quad \text{and}\quad \psi(\delta u)\le\psi(\delta)\quad\text{for}\quad u\ge 1,$$

one can easily verify that

$$\int_1^\infty |\tau''(u)|du = O(1),\quad \delta\to\infty. \tag{75}$$

Combining (67), (68), (74), and (75), we get

$$\left|\int_0^\infty \tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right| = O\left(\frac{1}{\delta\psi(\delta)}+\frac{1}{\delta^2\psi(\delta)}\int_1^\delta u\psi(u)du\right)\frac{1}{t^2}.$$

Hence,

$$\int_{|t|\ge\delta\pi/2} |\hat{\tau}_\delta(t)|\,dt = O\left(\frac{1}{\delta^2\psi(\delta)}+\frac{1}{\delta^3\psi(\delta)}\int_1^\delta u\psi(u)du\right),\quad \delta\to\infty.$$

This and relation (5) yield equality (62).

Theorem 1 is proved.

Corollary 1. *Suppose that the conditions of Theorem 1 are satisfied,*

$$\sin\frac{\beta\pi}{2}\ne 0,$$

and

$$\lim_{t\to\infty}\alpha(t)=\infty,$$

where $\alpha(t)$ *is defined by (30). Then the following asymptotic equality is true:*

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; B_\delta\right)_C = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \int\limits_{\delta}^{\infty} \frac{\psi(u)}{u} du + O\left(\psi(\delta)\right), \quad \delta \to \infty. \tag{76}$$

Examples of functions that satisfy the conditions of Corollary 1 are functions of the form

$$\psi(u) = \frac{1}{\ln^{\alpha}(u+K)}, \quad \alpha > 1, \quad K > 0.$$

Corollary 2. *Suppose that* ψ *belongs to the set* $\mathfrak{M}_0$,

$$\sin\frac{\beta\pi}{2} \neq 0,$$

the limit

$$\lim_{t\to\infty} \alpha(t)$$

exists, the function $u^2\psi(u)$ *is convex either upward or downward on* $[b,\infty)$, $b \geq 1$, *and*

$$\lim_{u\to\infty} u^2\psi(u) = \infty, \qquad \lim_{\delta\to\infty} \frac{1}{\delta^2\psi(\delta)} \int\limits_{1}^{\delta} u\psi(u)du = \infty.$$

Then the following asymptotic equality is true:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; B_\delta\right)_C = \frac{1}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta^2} \int\limits_{1}^{\delta} u\psi(u)du + O\left(\psi(\delta)\right), \quad \delta \to \infty. \tag{77}$$

Examples of functions that satisfy the conditions of Corollary 2 are functions of the form

$$\psi(u) = \frac{1}{u^2}\ln^{\alpha}(u+K), \quad K > 0, \quad \alpha > 0.$$

Corollary 3. *Suppose that* ψ *belongs to the set* $\mathfrak{M}_0$,

$$\sin\frac{\beta\pi}{2} \neq 0,$$

the function $u^2\psi(u)$ *is convex downward on* $[b,\infty)$, $b \geq 1$, *and*

$$\lim_{u\to\infty} u^2\psi(u) = K < \infty, \qquad \lim_{\delta\to\infty} \int\limits_{1}^{\delta} u\psi(u)du = \infty.$$

Then the following asymptotic equality is true:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; B_\delta\right)_C = \frac{1}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\frac{1}{\delta^2}\int_1^\delta u\psi(u)du + O\left(\frac{1}{\delta^2}\right), \quad \delta\to\infty. \tag{78}$$

Examples of functions that satisfy the conditions of Corollary 3 are the functions

$$\psi(u) = \frac{1}{u^2}\arctan u, \quad \psi(u) = \frac{1}{u^2}(K + e^{-u}), \quad \psi(u) = \frac{1}{u^2}\ln^\alpha(u+K), \quad K > 0, \ -1 \le \alpha \le 0.$$

In particular, if

$$\psi(u) = \frac{1}{u^2},$$

then relation (78) yields

$$\mathcal{E}\left(W^{2}_{\beta,\infty}; B_\delta\right)_C = \frac{1}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\frac{\ln\delta}{\delta^2} + O\left(\frac{1}{\delta^2}\right), \quad \delta\to\infty.$$

Note that, under the conditions of Corollaries 1–3, equalities (76)–(78) give a solution of the Kolmogorov–Nikol'skii problem for biharmonic Poisson integrals on the classes $C^{\psi}_{\beta,\infty}$ in the metric of the space C in the case where the rate of convergence of the functions $\psi(\cdot)$ to zero is low.

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