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NEVANLINNA CHARACTERISTICS AND DEFECTIVE VALUES OF THE WEIERSTRASS ZETA FUNCTION

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We determine the Nevanlinna characteristics of the Weierstrass zeta function and show that none of the values $a \in \overline{C}$ is exceptional in Nevanlinna's sense for this function.

In the present paper, we determine the Nevanlinna characteristics of the well-known Weierstrass zeta function $\zeta(z)$, which is closely related to the Weierstrass functions $\sigma(z)$ and $\wp(z)$ [1]. We also study the problem of defective values of the ζ-function. These problems can be investigated by using the asymptotic formulas from [2, 3], but we use a simpler method. These functions are often used in the investigation of elliptic functions. Note that $\zeta(z)$ is a meromorphic function with simple poles $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$ representable in the form

$$\zeta(z) = \frac{1}{z} + \sum_{m,n=-\infty}^{+\infty}{}' \left\{ \frac{1}{z-\Omega_{mn}} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2} \right\};$$

here, $\operatorname{Im}(\omega_2/\omega_1) > 0$, $m \in Z$, $n \in \mathbb{Z}$, and the prime near the sum sign means that the term with $m = 0$ and $n = 0$ is neglected. The functions $\sigma(z)$, $\zeta(z)$, and $\wp(z)$ are related by the equalities $\zeta(z) = \sigma'(z)/\sigma(z)$ and $\wp(z) = -\zeta'(z)$; furthermore, $\sigma(z)$ is an entire function with simple zeros Ω_{mn}, and $\wp(z)$ is a doubly periodic meromorphic function with poles of the second order at the indicated points, i.e., it is an elliptic function.

We use the main notions, facts, and standard notation of the theory of distribution of values of meromorphic functions (see [4]). Recall some of them.

The Nevanlinna characteristics of a meromorphic function f, $f \neq \text{const}$, are introduced by the equalities

$$m(r,f) := \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \left| f(re^{i\varphi}) \right| d\varphi,$$

$$N(r,f) := \int_0^r \frac{n(t,f)-n(0,f)}{t} dt + n(0,f) \ln r,$$

$$T(r,f) := m(r,f) + N(r,f),$$

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where $\ln^+ \alpha := \max\{0, \ln\alpha\}$, $\alpha > 0$, and $n(r,f)$ (also denoted by $n(r,\infty,f)$) is the number of poles of the function f in the disk $\{z \in \mathbb{C} \mid |z| \le r\}$, $r \ge 0$, counting multiplicities. If $a \in \mathbb{C}$, then the notation $n(r,a,f)$, $N(r,a,f)$, $m(r,a,f)$ is used instead of

$$n\left(r, \frac{1}{f-a}\right), \quad N\left(r, \frac{1}{f-a}\right), \quad m\left(r, \frac{1}{f-a}\right).$$

The Nevanlinna defect of a meromorphic function f at a point $a \in \overline{\mathbb{C}}$ is defined as follows:

$$\delta(a,f) := \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}.$$

If $\delta(a,f) > 0$, then a is called an exceptional (defective) value in Nevanlinna's sense for the meromorphic function f.

Theorem 1. *The following relations are true* $(r \to \infty)$:

$$N(r,\zeta) = \frac{\pi r^2}{2D} + O(r),$$

$$m(r,\zeta) = O(\ln r), \tag{1}$$

$$T(r,\zeta) = \frac{\pi r^2}{2D} + O(r), \tag{2}$$

where D is the area of the primitive period parallelogram of the function $\wp(z)$.

Proof. It is known [1, p. 420] that

$$N(r,\wp) = \frac{\pi r^2}{D} + O(r), \quad r \to \infty.$$

Therefore,

$$N(r,\zeta) = \frac{\pi r^2}{2D} + O(r), \quad r \to \infty.$$

Since $\zeta(z) = \sigma'(z)/\sigma(z)$, according to Theorem 1.3 in [4, p. 122] we get

$$m(r,\zeta) = m\left(r, \frac{\sigma'}{\sigma}\right) = O(\ln r), \quad r \to \infty.$$

Thus,

$$T(r,\zeta) = m(r,\zeta) + N(r,\zeta) = \frac{\pi r^2}{2D} + O(r), \quad r \to \infty .$$

The theorem is proved.

Theorem 2. *None of the values* $a \in \mathbb{C}$ *is exceptional in Nevanlinna's sense for the function* $\zeta(z)$.

Proof. It follows from (1) and (2) that

$$\delta(\infty,\zeta) = \liminf_{r\to\infty} \frac{m(r,\zeta)}{T(r,\zeta)} = 0 ,$$

i.e., the Nevanlinna defect of the ζ-function at the point ∞ is equal to zero.

It is known [1, p. 422] that

$$m(r,0,\wp) = O(r), \quad r \to \infty . \tag{3}$$

Using properties of the characteristic $m(r,a,f)$ of a meromorphic function f, Lemma 2.1 in [4, p. 129], estimates (1) and (3), and the equality $\zeta'(z) = -\wp(z)$, we conclude that the following relation holds for $a \in \mathbb{C}$, $a \neq 0$:

$$m(r,a,\zeta) \le m\left(r,\frac{\zeta}{\zeta'}\right) + O(\ln r) \le m\left(r,\frac{1}{\zeta'}\right) + m(r,\zeta) + O(\ln r) = m\left(r,\frac{1}{\wp}\right) + O(\ln r)$$

$$= m(r,0,\wp) + O(\ln r) = O(r), \quad r \to \infty .$$

It follows from (2) that

$$\delta(a,\zeta) = \liminf_{r\to\infty} \frac{m(r,a,\zeta)}{T(r,\zeta)} \le \lim_{r\to\infty} \frac{O(r)}{T(r,\zeta)} = 0 ,$$

i.e., the Nevanlinna defect of the function $\zeta(z)$ at an arbitrary point $a \in \mathbb{C}$, $a \neq 0$, is equal to zero, i.e., $\delta(a,\zeta) = 0$.

Consider the case $a = 0$. Using properties of the characteristic $m(r,a,f)$ of a meromorphic function f, estimate (3), and Theorem 1.3 in [4, p. 122], we get

$$m(r,0,\zeta) = m\left(r,\frac{1}{\zeta}\right) = m\left(r,\frac{\zeta'}{\zeta}\cdot\frac{1}{\zeta'}\right) \le m\left(r,\frac{\zeta'}{\zeta}\right) + m\left(r,\frac{1}{\zeta'}\right)$$

$$= m\left(r,\frac{1}{\wp}\right) + O(\ln r) = m(r,0,\wp) + O(\ln r) = O(r) + O(\ln r) = O(r), \quad r \to \infty .$$

Using equality (2), we obtain

$$\delta(0,\zeta) \;=\; \liminf_{r\to\infty} \frac{m(r,0,\zeta)}{T(r,\zeta)} \;\le\; \lim_{r\to\infty} \frac{O(r)}{T(r,\zeta)} \;=\; 0\,,$$

i.e., the Nevanlinna defect of the function $\zeta(z)$ at the point $a = 0$ is equal to zero $(\delta(0,\zeta) = 0)$.

The theorem is proved.

Remark. In particular, it follows from equality (2) that $\zeta(z)$ is a meromorphic function of order 2.

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