

# APPROXIMATION OF FUNCTIONS FROM THE CLASSES $C_{\beta, \infty}^{\psi}$ BY BIHARMONIC POISSON INTEGRALS

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We deduce asymptotic equalities for the upper bounds of deviations of biharmonic Poisson integrals on the classes of  $(\psi, \beta)$ -differentiable periodic functions in the uniform metric.

## 1. Statement of the Problem and Auxiliary Statements

Let  $L_1$  be a space of  $2\pi$ -periodic functions  $f(t)$  summable on  $(0, 2\pi)$  with norm

$$\|f\|_{L_1} = \|f\|_1 = \int_{-\pi}^{\pi} |f(t)| dt,$$

let  $L_{\infty}$  be a space of measurable and essentially bounded  $2\pi$ -periodic functions  $f(t)$  with norm  $\|f\|_{\infty} = \text{ess sup}_t |f(t)|$ , and let  $C$  be a space of continuous  $2\pi$ -periodic functions  $f(t)$  with norm  $\|f\|_C = \max_t |f(t)|$ .

For each function  $f \in L_1$ , we consider a function

$$B(\rho; f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) \left( \frac{1}{2} + \sum_{k=1}^{\infty} \left[ 1 + \frac{k}{2}(1-\rho^2) \right] \rho^k \cos kt \right) dt, \quad 0 \leq \rho < 1,$$

which is a solution (see, e.g., [1, p. 248]) of a biharmonic equation

$$\Delta^2 B = 0,$$

$$\Delta^2 B = \Delta(\Delta B), \quad \Delta = \frac{1}{\rho^2} \frac{\partial^2}{\partial x^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right).$$

We set  $\rho = e^{-1/\delta}$  and denote the biharmonic function  $B(\rho; f; x)$  by  $B_{\delta} = B_{\delta}(f; x)$ ,  $\delta > 0$ . It is called a biharmonic Poisson integral. In the present paper, we study the approximating properties of the biharmonic Poisson integral on the class of  $(\psi, \beta)$ -differentiable continuous functions.

Let  $f \in C$  and let  $a_k$  and  $b_k$  be its Fourier coefficients. If the sequence of real numbers  $\psi(k)$ ,  $k \in \mathbb{N}$ , and a fixed real number  $\beta$  are such that the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left( a_k \cos \left( kx + \frac{\pi\beta}{2} \right) + b_k \sin \left( kx + \frac{\pi\beta}{2} \right) \right)$$

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is the Fourier series of a function  $\varphi \in L_1$ , then  $\varphi(\cdot)$  is called the  $(\psi, \beta)$ -derivative of the function  $f(\cdot)$  in Stepanets' sense [2–4] and denoted by  $f_\beta^\psi(\cdot)$ . In this case, it is said that the function  $f(\cdot)$  belongs to the set  $C_\beta^\psi$ . If  $f \in C_\beta^\psi$  and  $f_\beta^\psi \in \mathfrak{M}$ ,  $\mathfrak{M} \subseteq L_1$ , then we say that  $f \in C_\beta^\psi \mathfrak{M}$ . Further, if  $\mathfrak{M}$  coincides with the unit ball of the space  $L_\infty$ , i.e.,

$$\mathfrak{M} = \left\{ f_\beta^\psi \in L_\infty : \operatorname{ess\,sup}_t |f_\beta^\psi(t)| \leq 1 \right\},$$

then the classes  $C_\beta^\psi \mathfrak{M}$  are denoted by  $C_{\beta,\infty}^\psi$ . For  $\psi(k) = k^{-r}$ ,  $r > 0$ , the classes  $C_{\beta,\infty}^\psi$  coincide with classes  $W_{\beta,\infty}^r$  introduced in [5] and  $f_\beta^\psi = f_\beta^{(r)}$  is the  $(r, \beta)$ -derivative in the Weil–Nagy sense. Moreover, if  $\beta = r$ ,  $r \in \mathbb{N}$ , then  $f_\beta^\psi$  is the derivative of the function  $f$  of order  $r$ . In this case,  $C_{\beta,\infty}^\psi$  are the well-known Sobolev classes  $W_\infty^r$ . In the case where  $\beta = r + 1$ ,  $r \in \mathbb{N}$ , the classes  $W_{\beta,\infty}^r$  coincide with the classes of conjugate functions  $\overline{W}_\infty^r$ .

It is convenient to treat the sequences  $\psi(k)$ ,  $k \in \mathbb{N}$ , specifying the classes  $C_{\beta,\infty}^\psi$  as restrictions to the set of natural numbers  $\mathbb{N}$  of certain functions  $\psi(t)$  of continuous argument  $t \geq 1$  running through a set  $\mathfrak{M}$ :

$$\mathfrak{M} := \left\{ \psi(t) : \psi(t) > 0, \psi(t_1) - 2\psi((t_1 + t_2)/2) + \psi(t_2) \geq 0 \right. \\ \left. \forall t_1, t_2 \in [1, \infty), \lim_{t \rightarrow \infty} \psi(t) = 0 \right\}.$$

Following Stepanets (see, e.g., [3, p. 93] or [4, p. 160]), every function  $\psi \in \mathfrak{M}$  is associated with the characteristics

$$\eta(t) = \eta(\psi; t) = \psi^{-1}(\psi(t)/2) \quad \text{and} \quad \mu(t) = \mu(\psi; t) = \frac{t}{\eta(t) - t}, \tag{1}$$

where  $\psi^{-1}$  is the function inverse to  $\psi$ . By using the function  $\mu(\psi; t)$ , we select subsets  $\mathfrak{M}_0$ ,  $\mathfrak{M}_C$ , and  $\mathfrak{M}_\infty$  of the set  $\mathfrak{M}$  as follows:

$$\mathfrak{M}_0 = \{ \psi \in \mathfrak{M} : 0 < \mu(\psi; t) \leq K \quad \forall t \geq 1 \}, \\ \mathfrak{M}_C = \{ \psi \in \mathfrak{M} : 0 < K_1 \leq \mu(\psi; t) \leq K_2 < \infty \quad \forall t \geq 1 \}, \\ \mathfrak{M}_\infty = \{ \psi \in \mathfrak{M} : 0 < K \leq \mu(\psi; t) < \infty \quad \forall t \geq 1 \},$$

where the constants  $K$ ,  $K_1$ , and  $K_2$  are, generally speaking, different in different relations and may depend on  $\psi$ .

Following Stepanets [4, p. 198], the problem of finding the asymptotic equalities for the quantities

$$\mathcal{E} \left( C_{\beta,\infty}^\psi; B_\delta \right)_C = \sup_{f \in C_{\beta,\infty}^\psi} \|f(\cdot) - B_\delta(f; \cdot)\|_C \tag{2}$$

as  $\delta \rightarrow \infty$  is called the Kolmogorov–Nikol'skii problem for the class  $C_{\beta,\infty}^\psi$  and the biharmonic Poisson integral in the uniform metric.

Note that the solution of the Kolmogorov–Nicol’skii problem in the class  $W_\infty^r$  was found by Kaniev [6] and Pych [7]. Moreover, Kaniev [8] also proved that the quantities  $\mathcal{E}(W_\infty^r; B_\delta)_C$  and  $\mathcal{E}(W_1^r; B_\delta)_1$  ( $W_1^r$  is the set of  $2\pi$ -periodic functions for which  $\|f^{(r)}(t)\|_1 \leq 1$ ) are equal, i.e., the estimates obtained for the uniform metric remain true for the integral metric. The approximating properties of biharmonic Poisson integrals in the classes of differentiable functions were also studied by Falaleev [9], Zhyhallo and Kharkevych [10, 11], Zastavnyi [12], and other mathematicians.

The aim of the present paper is to study the approximating properties of biharmonic Poisson integrals from the viewpoint of the Kolmogorov–Nicol’skii problem on the classes  $C_{\beta, \infty}^\psi$  of  $2\pi$ -periodic continuous functions  $f(\cdot)$  in the cases where these classes contain smooth and infinitely differentiable functions, i.e., for  $\psi \in \mathfrak{M}_C$  and  $\psi \in \mathfrak{M}_\infty$ .

For the biharmonic Poisson integral, we set

$$\tau(u) = \tau_\delta(u; \psi) = \begin{cases} (1 - [1 + \gamma u] e^{-u}) \frac{\psi(1)}{\psi(\delta)}, & 0 \leq u \leq \frac{1}{\delta}, \\ (1 - [1 + \gamma u] e^{-u}) \frac{\psi(\delta u)}{\psi(\delta)}, & u \geq \frac{1}{\delta}, \end{cases} \tag{3}$$

where  $\gamma = \gamma(\delta) = \frac{\delta}{2}(1 - e^{-2/\delta})$  and  $\psi(\cdot)$  is a function defined and continuous for  $u \geq 1$ . Repeating the Stepanets reasoning from [4, p. 183], we can show that if the Fourier transform

$$\hat{\tau}(t) = \hat{\tau}_\delta(t) = \frac{1}{\pi} \int_0^\infty \tau(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \tag{4}$$

of the function  $\tau(\cdot)$  given by relation (3) is summable on the entire real axis, i.e., the integral

$$A(\tau) = \int_{-\infty}^\infty |\hat{\tau}_\delta(t)| dt, \tag{5}$$

is convergent, then, for any  $f \in C_{\beta, \infty}^\psi$ , the equality

$$f(x) - B_\delta(f; x) = \psi(\delta) \int_{-\infty}^{+\infty} f_\beta^\psi\left(x + \frac{t}{\delta}\right) \hat{\tau}_\delta(t) dt, \quad \delta > 0, \tag{6}$$

holds at any point  $x \in \mathbb{R}$ . Thus, by using the integral representation (6), we arrive at the following expression for the quantity (2):

$$\mathcal{E}\left(C_{\beta, \infty}^\psi; B_\delta\right)_C = \sup_{f \in C_{\beta, \infty}^\psi} \left\| \psi(\delta) \int_{-\infty}^{+\infty} f_\beta^\psi\left(x + \frac{t}{\delta}\right) \hat{\tau}(t) dt \right\|_C. \tag{7}$$

**2. Asymptotic Equalities for the Upper Bounds of Deviations of the Biharmonic Poisson Integrals from the Functions of the Classes  $C_{\beta, \infty}^{\psi}$**

The following assertion is true:

**Theorem 1.** *Let  $\psi$  belong to  $\mathfrak{M}_C$ , let the function  $g(u) = u^2\psi(u)$  be convex downward on  $[b, \infty)$ ,  $b \geq 1$ , and let*

$$\int_1^{\infty} \frac{g(u)}{u} du < \infty. \tag{8}$$

Then the following asymptotic equality is true as  $\delta \rightarrow \infty$  :

$$\mathcal{E} \left( C_{\beta, \infty}^{\psi}; B_{\delta} \right)_C = \frac{1}{\delta^2} \sup_{f \in C_{\beta, \infty}^{\psi}} \left\| \frac{f_0^{(2)}(x)}{2} + f_0^{(1)}(x) \right\|_C + O \left( \frac{1}{\delta^3} \int_1^{\delta} t^2 \psi(t) dt + \frac{1}{\delta^2} \int_{\delta}^{\infty} t \psi(t) dt \right), \tag{9}$$

where  $f_0^{(1)}(\cdot)$  and  $f_0^{(2)}(\cdot)$  are the  $(\psi, \beta)$ -derivatives of the function  $f(\cdot)$  for  $\beta = 0$  and  $\psi(t) = \frac{1}{t}$  and  $\psi(t) = \frac{1}{t^2}$ , respectively.

**Proof.** We represent the function  $\tau(u)$  defined by relation (3) in the form of the sum of functions of this sort  $\varphi(u)$  and  $\nu(u)$  :

$$\varphi(u) = \begin{cases} \left( \frac{u^2}{2} + \frac{u}{\delta} \right) \frac{\psi(1)}{\psi(\delta)}, & 0 \leq u < \frac{1}{\delta}, \\ \left( \frac{u^2}{2} + \frac{u}{\delta} \right) \frac{\psi(\delta u)}{\psi(\delta)}, & u \geq \frac{1}{\delta}, \end{cases} \tag{10}$$

$$\nu(u) = \begin{cases} \left( 1 - [1 + \gamma u] e^{-u} - \frac{u^2}{2} - \frac{u}{\delta} \right) \frac{\psi(1)}{\psi(\delta)}, & 0 \leq u \leq \frac{1}{\delta}, \\ \left( 1 - [1 + \gamma u] e^{-u} - \frac{u^2}{2} - \frac{u}{\delta} \right) \frac{\psi(\delta u)}{\psi(\delta)}, & u \geq \frac{1}{\delta}. \end{cases} \tag{11}$$

By  $\hat{\varphi}(\cdot)$  and  $\hat{\nu}(\cdot)$  we denote the Fourier transforms of the functions  $\varphi$  and  $\nu$ , respectively:

$$\hat{\varphi}(t) = \hat{\varphi}_{\delta}(t) = \frac{1}{\pi} \int_0^{\infty} \varphi(u) \cos \left( ut + \frac{\beta\pi}{2} \right) du, \tag{12}$$

$$\hat{\nu}(t) = \hat{\nu}_{\delta}(t) = \frac{1}{\pi} \int_0^{\infty} \nu(u) \cos \left( ut + \frac{\beta\pi}{2} \right) du. \tag{13}$$

Further, by using Theorem 1 in [13], we show that the Fourier transforms  $\hat{\varphi}(\cdot)$  and  $\hat{\nu}(\cdot)$  are summable on the entire real axis.

To show that the Fourier transform  $\hat{\varphi}(\cdot)$  is summable on the entire real axis, it is necessary to show that the integral

$$A(\varphi) = \int_{-\infty}^{\infty} |\hat{\varphi}_{\delta}(t)| dt \tag{14}$$

is convergent. To this end, by Theorem 1 from [13, p.24], it suffices to prove the convergence of the following integrals:

$$\int_0^{1/2} u |d\varphi'(u)|, \quad \int_{1/2}^{\infty} |u - 1| |d\varphi'(u)|,$$

$$\left| \sin \frac{\beta\pi}{2} \right| \int_0^{\infty} \frac{|\varphi(u)|}{u} du, \quad \int_0^1 \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du.$$

It follows from relation (10) that

$$d\varphi'(u) = \frac{\psi(1)}{\psi(\delta)} du, \quad u \in \left[0, \frac{1}{\delta}\right).$$

Therefore,

$$\int_0^{1/\delta} u |d\varphi'(u)| = \frac{\psi(1)}{2\delta^2\psi(\delta)}. \tag{15}$$

By using the inequalities

$$\int_{1/\delta}^{1/2} u |d\varphi'(u)| \leq \int_{1/\delta}^{\infty} u |d\varphi'(u)| \quad \text{and} \quad \int_{1/2}^{\infty} |u - 1| |d\varphi'(u)| \leq \int_{1/\delta}^{\infty} u |d\varphi'(u)|,$$

we arrive at the following estimate:

$$\int_{1/\delta}^{\infty} u |d\varphi'(u)| \tag{16}$$

in each interval  $\left[\frac{1}{\delta}, \frac{b}{\delta}\right)$  and  $\left[\frac{b}{\delta}, \infty\right)$  (for  $\delta > 2b$ ).

It follows from relation (10) for  $u \geq \frac{1}{\delta}$  that

$$d\varphi'(u) = \left( \psi(\delta u) + 2 \left(u + \frac{1}{\delta}\right) \delta \psi'(\delta u) + \left(\frac{u^2}{2} + \frac{u}{\delta}\right) \delta^2 \psi''(\delta u) \right) \frac{du}{\psi(\delta)}. \tag{17}$$

In view of the fact that the function  $\psi(u)$  is convex downward and decreasing for  $u \geq 1$ , this yields

$$\int_{1/\delta}^{b/\delta} u |d\varphi'(u)| \leq \frac{1}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \left(\frac{u^3}{2} + \frac{u^2}{\delta}\right) \delta^2 \psi''(\delta u) du + \frac{2}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \left(u^2 + \frac{u}{\delta}\right) \delta |\psi'(\delta u)| du + \frac{1}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u \psi(\delta u) du. \tag{18}$$

Since  $\psi(\delta u) \leq \psi(1)$  for  $u \in \left[\frac{1}{\delta}, \frac{b}{\delta}\right)$ , we get

$$\frac{1}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u \psi(\delta u) du \leq \frac{\psi(1)}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u du = \frac{K}{\delta^2 \psi(\delta)}.$$

Integrating the first and second integrals on the right-hand side of inequality (18) by parts, we find

$$\int_{1/\delta}^{b/\delta} u |d\varphi'(u)| \leq \frac{K_1}{\delta^2 \psi(\delta)}. \tag{19}$$

To estimate integral (16) in the interval  $\left[\frac{b}{\delta}, \infty\right)$ , we use the relations

$$\lim_{u \rightarrow \infty} u^2 \psi(u) = 0, \tag{20}$$

$$\lim_{u \rightarrow \infty} u^3 \psi'(u) = 0. \tag{21}$$

We now prove these relations. Indeed, since the function  $g(u) = u^2 \psi(u)$  is convex downward for  $u \geq b \geq 1$ , the following cases are possible: either  $\lim_{u \rightarrow \infty} g(u) = 0$ , or  $\lim_{u \rightarrow \infty} g(u) = K > 0$ , or  $\lim_{u \rightarrow \infty} g(u) = \infty$ .

Let  $\lim_{u \rightarrow \infty} g(u) = K > 0$ . Then there exists  $0 < K_1 < K$  such that  $g(u) > K_1$  for all  $u \geq 1$  and, hence,  $\psi(u) > \frac{K_1}{u^2}$ . However, this contradicts the fact that, according to condition (8), the function  $u\psi(u)$  is summable on  $[1, \infty)$ .

Now let  $\lim_{u \rightarrow \infty} g(u) = \infty$ , i.e., for any  $M > 0$ , there exists  $N > 0$  such that the inequality  $g(u) > M$  holds for all  $u > N$ . Then

$$\int_1^x u \psi(u) du = \int_1^N u \psi(u) du + \int_N^x \frac{g(u)}{u} du > K_2 + \int_N^x \frac{M}{u} du = K_2 + M(\ln x - \ln N),$$

which also contradicts the condition of summability of the function  $u\psi(u)$  in the interval  $[1, \infty)$ . This enables us to conclude that relation (20) is true.

We now prove (21). The function  $g'(u)$  is summable on  $[1, \infty)$ . Then

$$\lim_{u \rightarrow \infty} \int_{u/2}^u u/2^u g'(x) dx = 0.$$

Since, for  $u \geq b \geq 1$ , the function  $g(u)$  is convex downward, the function  $(-g'(u))$ , for  $u \geq b$ , does not increase and, hence,

$$-\int_{u/2}^u g'(x) dx > -\left(u - \frac{u}{2}\right) (2u\psi(u) + u^2\psi'(u)) = -\frac{1}{2} (2u^2\psi(u) + u^3\psi'(u)).$$

This and (20) imply the validity of relation (21).

In view of (17), for any function  $\psi(\cdot) \in \mathfrak{M}$ , we find

$$\begin{aligned} & \int_{b/\delta}^{\infty} u |d\varphi'(u)| \\ & \leq \frac{1}{\psi(\delta)} \int_{b/\delta}^{\infty} \left(\frac{u^3}{2} + \frac{u^2}{\delta}\right) \delta^2 \psi''(\delta u) du + \frac{2}{\psi(\delta)} \int_{b/\delta}^{\infty} \left(u^2 + \frac{u}{\delta}\right) \delta |\psi'(\delta u)| du + \frac{1}{\psi(\delta)} \int_{b/\delta}^{\infty} u \psi(\delta u) du. \end{aligned} \quad (22)$$

Integrating the first and second integrals on the right-hand side of inequality (22) by parts and taking into account relations (20), (21), and (8), we obtain

$$\int_{b/\delta}^{\infty} u |d\varphi'(u)| \leq \frac{K_2}{\delta^2 \psi(\delta)}. \quad (23)$$

It follows from relations (15), (19), and (23) that

$$\int_0^{1/2} u |d\varphi'(u)| = O\left(\frac{1}{\delta^2 \psi(\delta)}\right) \quad \text{and} \quad \int_{1/2}^{\infty} |u - 1| |d\varphi'(u)| = O\left(\frac{1}{\delta^2 \psi(\delta)}\right) \quad \text{as } \delta \rightarrow \infty, \quad (24)$$

In view of relations (10) and (8), we find

$$\int_0^{\infty} \frac{|\varphi(u)|}{u} du = \frac{\psi(1)}{\psi(\delta)} \int_0^{1/\delta} \left(\frac{u}{2} + \frac{1}{\delta}\right) du + \frac{1}{\psi(\delta)} \int_{1/\delta}^{\infty} \left(\frac{u}{2} + \frac{1}{\delta}\right) \psi(\delta u) du \leq \frac{K}{\delta^2 \psi(\delta)}.$$

Finally, we estimate the integral

$$\int_0^1 \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = \int_0^{1-1/\delta} \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du + \int_{1-1/\delta}^1 \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du. \quad (25)$$

Relation (10) can be represented in the form

$$\varphi(u) = \begin{cases} (1 - \lambda_{\delta,1}(u)) \frac{\psi(1)}{\psi(\delta)}, & 0 \leq u \leq \frac{1}{\delta}, \\ (1 - \lambda_{\delta,1}(u)) \frac{\psi(\delta u)}{\psi(\delta)}, & u \geq \frac{1}{\delta}, \end{cases} \tag{26}$$

where  $\lambda_{\delta,1}(u) = 1 - \frac{u^2}{2} - \frac{u}{\delta}$ . By using relation (26), we get

$$\varphi(1 - u) = \begin{cases} (1 - \lambda_{\delta,1}(1 - u)) \frac{\psi(1)}{\psi(\delta)}, & 1 - \frac{1}{\delta} \leq u \leq 1, \\ (1 - \lambda_{\delta,1}(1 - u)) \frac{\psi(\delta(1 - u))}{\psi(\delta)}, & u \leq 1 - \frac{1}{\delta}, \end{cases} \tag{27}$$

$$\varphi(1 + u) = \begin{cases} (1 - \lambda_{\delta,1}(1 + u)) \frac{\psi(1)}{\psi(\delta)}, & -1 \leq u \leq \frac{1}{\delta} - 1, \\ (1 - \lambda_{\delta,1}(1 + u)) \frac{\psi(\delta(1 + u))}{\psi(\delta)}, & u \geq \frac{1}{\delta} - 1. \end{cases} \tag{28}$$

We now estimate the first term on the right-hand side of (25) by adding and subtracting the expression  $\lambda_{\delta,1}(1 - u) - \lambda_{\delta,1}(1 + u)$  under the modulus sign in the integrand. Thus, we get

$$\begin{aligned} \int_0^{1-1/\delta} \frac{|\varphi(1 - u) - \varphi(1 + u)|}{u} du &\leq \int_0^{1-1/\delta} \frac{|\lambda_{\delta,1}(1 - u) - \lambda_{\delta,1}(1 + u)|}{u} du \\ &+ \int_0^{1-1/\delta} \frac{|\varphi(1 - u) - \varphi(1 + u) + \lambda_{\delta,1}(1 - u) - \lambda_{\delta,1}(1 + u)|}{u} du. \end{aligned} \tag{29}$$

It is easy to see that the first integral on the right-hand side of inequality (29) satisfies the estimate

$$\int_0^{1-1/\delta} \frac{|\lambda_{\delta,1}(1 - u) - \lambda_{\delta,1}(1 + u)|}{u} du = O(1). \tag{30}$$

Since relations (27) and (28) are true, for  $u \in \left[0, 1 - \frac{1}{\delta}\right]$ , we obtain

$$\lambda_{\delta,1}(1 - u) = 1 - \frac{\psi(\delta)}{\psi(\delta(1 - u))} \varphi(1 - u), \quad \lambda_{\delta,1}(1 + u) = 1 - \frac{\psi(\delta)}{\psi(\delta(1 + u))} \varphi(1 + u).$$



Then

$$\begin{aligned} & \int_0^{1-1/\delta} \frac{|\varphi(1-u) - \varphi(1+u) + (\lambda_{\delta,1}(1-u) - \lambda_{\delta,1}(1+u))|}{u} du \\ & \leq \int_0^{1-1/\delta} |\varphi(1-u)| \left| 1 - \frac{\psi(\delta)}{\psi(\delta(1-u))} \right| \frac{du}{u} + \int_0^{1-1/\delta} |\varphi(1+u)| \left| 1 - \frac{\psi(\delta)}{\psi(\delta(1+u))} \right| \frac{du}{u}. \end{aligned} \tag{31}$$

The function  $\varphi(\cdot)$  satisfies the conditions of Lemma 2 in [13]. Therefore,

$$|\varphi(u)| \leq |\varphi(0)| + |\varphi(1)| + \int_0^{1/2} u |d\varphi'(u)| + \int_{1/2}^{\infty} |u-1| |d\varphi'(u)| := H(\varphi).$$

Hence, we find

$$\begin{aligned} & \int_0^{1-1/\delta} \frac{|\varphi(1-u) - \varphi(1+u) + (\lambda_{\delta,1}(1-u) - \lambda_{\delta,1}(1+u))|}{u} du \\ & = H(\varphi) O \left( \int_0^{1-1/\delta} \frac{|\psi(\delta(1-u)) - \psi(\delta)|}{u\psi(\delta(1-u))} du + \int_0^{1-1/\delta} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u\psi(\delta(1+u))} du \right). \end{aligned} \tag{32}$$

By using relation (10) and estimates (24), we get

$$H(\varphi) = O \left( 1 + \frac{1}{\delta^2 \psi(\delta)} \right), \quad \delta \rightarrow \infty. \tag{33}$$

It is easy to see that, for  $\psi \in \mathfrak{M}_C$ , the integrals on the right-hand side of (32) satisfy the following estimates:

$$\int_0^{1-1/\delta} \frac{|\psi(\delta(1-u)) - \psi(\delta)|}{u\psi(\delta(1-u))} du = O(1) \quad \text{and} \quad \int_0^{1-1/\delta} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u\psi(\delta(1+u))} du = O(1) \quad \text{as } \delta \rightarrow \infty,$$

whence, by combining relations (29)–(33), in view of (20), we obtain

$$\int_0^{1-1/\delta} \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = O \left( \frac{1}{\delta^2 \psi(\delta)} \right).$$

Reasoning as above, one can easily show that the second term on the right-hand side of (25) admits the same estimate. Hence,

$$\int_0^1 |\varphi(1-u) - \varphi(1+u)| \frac{du}{u} = O\left(\frac{1}{\delta^2 \psi(\delta)}\right), \quad \delta \rightarrow \infty.$$

Thus, by Theorem 1 in [13], integral (14) is convergent.

The summability of the transform  $\hat{v}(t)$  given by relation (13) on the entire real axis follows from convergence of the integral

$$A(v) = \int_{-\infty}^{\infty} |\hat{v}_\delta(t)| dt. \tag{34}$$

In order that the integral  $A(v)$  be convergent, it is necessary and sufficient (see Theorem 1 [13, p.24]) that the following integrals be convergent:

$$\int_0^{1/2} u |dv'(u)|, \quad \int_{1/2}^{\infty} |u-1| |dv'(u)|, \tag{35}$$

$$\left| \sin \frac{\beta\pi}{2} \right| \int_0^{\infty} \frac{|v(u)|}{u} du, \quad \int_0^1 \frac{|v(1-u) - v(1+u)|}{u} du, \tag{36}$$

where  $v(u)$  is the function given by relation (11) defined and continuous for all  $u \geq 0$ .

We now estimate the first integral in (35) on each segment  $\left[0, \frac{1}{\delta}\right]$ ,  $\left[\frac{1}{\delta}, \frac{b}{\delta}\right]$ , and  $\left[\frac{b}{\delta}, \frac{1}{2}\right]$ ,  $\delta > 2b$ . Denote

$$\bar{v}(u) := 1 - e^{-u} - \gamma u e^{-u} - \frac{u^2}{2} - \frac{u}{\delta}. \tag{37}$$

By using (37), we represent the function  $v(u)$  of the form (11) on the segment  $\left[0, \frac{1}{\delta}\right]$  as follows:

$$v(u) = \bar{v}(u) \frac{\psi(1)}{\psi(\delta)}.$$

Relation (37) now implies that

$$\bar{v}'(u) = e^{-u} - \gamma e^{-u} + \gamma u e^{-u} - u - \frac{1}{\delta},$$

$$\bar{v}''(u) = -e^{-u} + 2\gamma e^{-u} - \gamma u e^{-u} - 1,$$

$$\bar{v}(0) = 0, \quad \bar{v}'(0) = 1 - \gamma - \frac{1}{\delta} < 0,$$

whence, in view of the fact that

$$-1 + 2\gamma - \gamma u < e^u, \quad u \in [0, \infty),$$

we arrive at the inequalities

$$\bar{v}(u) \leq 0, \quad \bar{v}'(u) < 0, \quad \bar{v}''(u) < 0, \quad u \geq 0. \tag{38}$$

Thus, for the function  $v(\cdot)$  defined by relation (11), in view of (37) and the third inequality in (38), we obtain

$$v''(u) = \bar{v}''(u) \frac{\psi(1)}{\psi(\delta)} < 0, \quad u \in \left[0, \frac{1}{\delta}\right]. \tag{39}$$

Therefore,

$$\int_0^{1/\delta} u |dv'(u)| = - \int_0^{1/\delta} u dv'(u) = \bar{v} \left(\frac{1}{\delta}\right) \frac{\psi(1)}{\psi(\delta)} - \frac{1}{\delta} \bar{v}' \left(\frac{1}{\delta}\right) \frac{\psi(1)}{\psi(\delta)}.$$

By virtue of the relations

$$|\bar{v}(u)| < \frac{2}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{u^3}{2}, \quad |\bar{v}'(u)| < \frac{2}{3\delta^2} + \frac{2}{\delta}u + \frac{3}{2}u^2, \quad u \geq 0, \tag{40}$$

we find

$$\int_0^{1/\delta} u |dv'(u)| = O\left(\frac{1}{\delta^3\psi(\delta)}\right). \tag{41}$$

Further, we estimate the first integral in (35) on the interval  $\left[\frac{1}{\delta}, \frac{b}{\delta}\right]$ ,  $\delta > 2b$ . By using the equality

$$v''(u) = \bar{v}''(u) \frac{\psi(\delta u)}{\psi(\delta)} + 2\delta \bar{v}'(u) \frac{\psi'(\delta u)}{\psi(\delta)} + \delta^2 \bar{v}(u) \frac{\psi''(\delta u)}{\psi(\delta)}, \tag{42}$$

we get

$$\int_{1/\delta}^{b/\delta} u |dv'(u)| \leq \frac{1}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u |\bar{v}''(u)| \psi(\delta u) du + \frac{2\delta}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u |\bar{v}'(u)| |\psi'(\delta u)| du + \frac{\delta^2}{\psi(\delta)} \int_{1/\delta}^{b/\delta} u |\bar{v}(u)| \psi''(\delta u) du.$$

By using inequalities (40) once again, in view of the estimate  $|\bar{v}''(u)| < \frac{2}{\delta} + 3u$ ,  $u \geq 0$ , as a result of the integration by parts, we obtain

$$\int_{1/\delta}^{b/\delta} u |dv'(u)| \leq \frac{K_2}{\delta^3\psi(\delta)}. \tag{43}$$

Further, we show that if the function  $u^2\psi(u)$  is convex downward for  $u \geq b$ ,  $b \geq 1$ , then the following inequality holds:

$$dv'(u) \leq 0, \quad u \geq b/\delta, \tag{44}$$

Consider a function

$$\tilde{v}(u) = \frac{1}{u^2} - \frac{e^{-u}}{u^2} - \gamma \frac{e^{-u}}{u} - \frac{1}{2} - \frac{1}{u\delta}.$$

We have

$$\begin{aligned} \tilde{v}(u) &= \frac{\bar{v}(u)}{u^2}, \quad \gamma > 1 - \frac{1}{\delta}, \\ \tilde{v}'(u) &= -\frac{2}{u^3} + \frac{2e^{-u}}{u^3} + \frac{e^{-u}}{u^2} + \gamma \frac{e^{-u}}{u^2} + \gamma \frac{e^{-u}}{u} + \frac{1}{u^2\delta} \\ &= \frac{1}{u^3} \left( -2 + 2e^{-u} + (1 + \gamma)ue^{-u} + \gamma u^2e^{-u} + \frac{u}{\delta} \right), \\ \tilde{v}''(u) &= \frac{6}{u^4} - \frac{6e^{-u}}{u^4} - \frac{4e^{-u}}{u^3} - \frac{e^{-u}}{u^2} - \gamma \frac{2e^{-u}}{u^3} - 2\gamma \frac{e^{-u}}{u^2} - \gamma \frac{e^{-u}}{u} - \frac{2}{u^3\delta} \\ &= \frac{1}{u^4} \left( 6 - 6e^{-u} - (4 + 2\gamma)ue^{-u} - (1 + 2\gamma)u^2e^{-u} - \gamma u^3e^{-u} - \frac{2u}{\delta} \right). \end{aligned}$$

Thus, in view of the inequality  $e^{-u} \geq 1 - u$ , we find

$$\begin{aligned} \tilde{v}(u) &< 0, \\ \tilde{v}'(u) &> \frac{1}{u^3} \left( -2 + 2 - 2u + \left( 1 + 1 - \frac{1}{\delta} \right) (u - u^2) + \gamma u^2e^{-u} + \frac{u}{\delta} \right) = \frac{1}{u^3} \left( \frac{u^2}{\delta} + \gamma u^2e^{-u} \right) > 0, \\ \tilde{v}''(u) &< \frac{1}{u^4} \left( 6 - 6 + 6u - \left( 4 + 2 - \frac{2}{\delta} \right) (u - u^2) - (1 + 2\gamma)u^2e^{-u} - \gamma u^3e^{-u} - \frac{2u}{\delta} \right) \\ &= \frac{1}{u^4} \left( -\frac{2u^2}{\delta} - (1 + 2\gamma)u^2e^{-u} - \gamma u^3e^{-u} \right) < 0. \end{aligned}$$

Finally, since  $g(u) > 0$ ,  $g'(u) < 0$ , and  $g''(u) > 0$  for  $u \geq b$ ,  $b \geq 1$ , we conclude that

$$v''(u) = \left( \frac{1}{\delta^2} \tilde{v}(u)g(\delta u) \right)'' = \frac{1}{\delta^2} \tilde{v}''(u)g(\delta u) + \frac{2}{\delta} \tilde{v}'(u)g'(\delta u) + \tilde{v}(u)g''(\delta u) < 0$$

for  $u \geq \frac{b}{\delta}$ .

Further, we use the following assertions:

**Proposition 1** [4, p. 161]. *A function  $\psi \in \mathfrak{M}$  belongs to  $\mathfrak{M}_C$  if and only if the quantity*

$$\alpha(t) = \frac{\psi(t)}{t |\psi'(t)|}, \quad \psi'(t) = \psi'(t + 0),$$

*satisfies the condition  $0 < K_1 \leq \alpha(t) \leq K_2 \quad \forall t \geq 1$ .*

**Proposition 2** [4, p. 175]. *In order that the function  $\psi \in \mathfrak{M}$  belong to  $\mathfrak{M}_0$ , it is necessary and sufficient that, for any fixed number  $c > 1$ , there exist a constant  $K$  such that the inequality*

$$\frac{\psi(t)}{\psi(ct)} \leq K$$

*holds for all  $t \geq 1$ .*

In view of relations (44) and (40) and Propositions 1 and 2, for functions  $\psi(\cdot)$  from the class  $\mathfrak{M}_C$ , we obtain

$$\int_{b/\delta}^{1/2} u |dv'(u)| = - \int_{b/\delta}^{1/2} u dv'(u) = -\frac{1}{2}v' \left( \frac{1}{2} \right) + \frac{b}{\delta}v' \left( \frac{b}{\delta} \right) + v \left( \frac{1}{2} \right) - v \left( \frac{b}{\delta} \right) \leq K_1 + \frac{K_2}{\delta^3 \psi(\delta)}. \quad (45)$$

Combining relations (41), (43), and (45), we arrive at the estimate

$$\int_0^{1/2} u |dv'(u)| = O \left( 1 + \frac{1}{\delta^3 \psi(\delta)} \right). \quad (46)$$

In view of relations (20) and (21) and Propositions 1 and 2, we readily conclude that the second integral in (35) satisfies the following estimate as  $\delta \rightarrow \infty$ :

$$\int_{1/2}^{\infty} |u - 1| |dv'(u)| = O(1). \quad (47)$$

We now estimate the first integral in (36) on each interval  $\left[ 0, \frac{1}{\delta} \right]$ ,  $\left[ \frac{1}{\delta}, 1 \right]$ , and  $\left[ \frac{1}{\delta}, \infty \right)$ . By using the first inequality in (38), we conclude that  $v(u) \leq 0$  for  $\left[ 0, \frac{1}{\delta} \right]$ . Hence, in view of the fact that

$$e^{-u} \leq 1 - u + \frac{u^2}{2}, \quad u \geq 0, \quad (48)$$

we find

$$\begin{aligned} \int_0^{1/\delta} \frac{|v(u)|}{u} du &= \frac{\psi(1)}{\psi(\delta)} \int_0^{1/\delta} \left( -1 + e^{-u} + \gamma u e^{-u} + \frac{u^2}{2} + \frac{u}{\delta} \right) \frac{du}{u} \\ &\leq \frac{\psi(1)}{\psi(\delta)} \int_0^{1/\delta} \left( \left( -1 + \gamma + \frac{1}{\delta} \right) + (1 - \gamma)u + \frac{\gamma}{2}u^2 \right) du, \end{aligned}$$

whence, by virtue of the inequalities

$$\gamma < 1, \quad 1 - \gamma < \frac{1}{\delta}, \tag{49}$$

$$-1 + \gamma + \frac{1}{\delta} < \frac{2}{3\delta^2}, \tag{50}$$

we get

$$\int_0^{1/\delta} \frac{|v(u)|}{u} du = O\left(\frac{1}{\delta^3 \psi(\delta)}\right), \quad \delta \rightarrow \infty. \tag{51}$$

By using inequalities (48)–(50) once again, we conclude that

$$\begin{aligned} \int_{1/\delta}^1 \frac{|v(u)|}{u} du &\leq \int_{1/\delta}^1 \frac{\psi(\delta u)}{\psi(\delta)} \left( \frac{1}{\delta} + \gamma - 1 + (1 - \gamma)u + \frac{\gamma}{2}u^2 \right) du \\ &\leq \frac{K_1}{\delta^3 \psi(\delta)} \int_1^\delta \psi(u) du + \frac{K_2}{\delta^3 \psi(\delta)} \int_1^\delta u \psi(u) du + \frac{K_3}{\delta^3 \psi(\delta)} \int_1^\delta u^2 \psi(u) du \\ &= O\left(\frac{1}{\delta^3 \psi(\delta)} \int_1^\delta u^2 \psi(u) du\right), \quad \delta \rightarrow \infty, \end{aligned} \tag{52}$$

$$\begin{aligned} \int_1^\infty \frac{|v(u)|}{u} du &= \frac{1}{\psi(\delta)} \int_1^\infty \psi(\delta u) \left( \frac{e^{-u} - 1}{u} + \gamma e^{-u} + \frac{u}{2} + \frac{1}{\delta} \right) du \\ &\leq \frac{1}{\psi(\delta)} \int_1^\infty \psi(\delta u) \left( -1 + \frac{u}{2} + \gamma + \frac{u}{2} + \frac{1}{\delta} \right) du = O\left(\frac{1}{\delta^2 \psi(\delta)} \int_\delta^\infty u \psi(u) du\right). \end{aligned} \tag{53}$$

Combining (51)–(53) and taking into account the fact that

$$\int_1^\delta u^2 \psi(u) du \geq K,$$

we arrive at the following estimate for the first integral in (36):

$$\int_0^{\infty} \frac{|v(u)|}{u} du = O \left( \frac{1}{\delta^3 \psi(\delta)} \int_1^{\delta} u^2 \psi(u) du + \frac{1}{\delta^2 \psi(\delta)} \int_{\delta}^{\infty} u \psi(u) du \right). \tag{54}$$

We now estimate the second integral in (36) on the segments  $[0, 1 - 1/\delta]$  and  $[1 - 1/\delta, 1]$ . Denote

$$\lambda_{\delta,2}(u) = [1 + \gamma u] e^{-u} + \frac{u^2}{2} + \frac{u}{\delta}.$$

This enables us to represent the function  $v(\cdot)$  of the form (11) in the form (26). Further, for the function  $v(\cdot)$ , we use the same reasoning as in deducing relations (27)–(32) and show that

$$\int_0^1 |v(1-u) - v(1+u)| \frac{du}{u} = \int_0^1 |\lambda_{\delta,2}(1-u) - \lambda_{\delta,2}(1+u)| \frac{du}{u} + O(H(v)), \tag{55}$$

where

$$H(v) := |v(0)| + |v(1)| + \int_0^{1/2} u |dv'(u)| + \int_{1/2}^{\infty} |u-1| |dv'(u)|.$$

According to (11), (46), and (47), the following estimate is true for the quantity  $H(v)$ :

$$H(v) = O \left( 1 + \frac{1}{\delta^3 \psi(\delta)} \right), \quad \delta \rightarrow \infty. \tag{56}$$

In addition,

$$\begin{aligned} & \int_0^1 \left| \frac{\lambda_{\delta,2}(1-u) - \lambda_{\delta,2}(1+u)}{u} \right| du \\ &= \int_0^1 \left| \frac{\gamma + 1}{e} \frac{e^u - e^{-u}}{u} - \frac{\gamma}{e} (e^u + e^{-u}) + 2 \left( 1 + \frac{1}{\delta} \right) \right| du = O(1), \quad \delta \rightarrow \infty. \end{aligned} \tag{57}$$

Comparing (55)–(57), we conclude that

$$\int_0^1 \frac{|v(1-u) - v(1+u)|}{u} du = O \left( 1 + \frac{1}{\delta^3 \psi(\delta)} \right) \quad \text{as } \delta \rightarrow \infty. \tag{58}$$

Therefore, by Theorem 1 in [13], integral (34) is also convergent.

Thus, it is shown that, under the conditions of Theorem 1, the integral  $A(\tau)$  of the form (5) is convergent and, hence, the Fourier transform  $\hat{\tau}(t)$  of the function  $\tau(u) = \varphi(u) + \nu(u)$  is summable on the entire real axis. Therefore, for every  $f \in C_{\beta, \infty}^\psi$ , equality (6) is true at any point  $x \in \mathbb{R}$ . In view of (34), we rewrite quantity (7) in the form

$$\begin{aligned} \mathcal{E} \left( C_{\beta, \infty}^\psi; B_\delta \right)_C &= \sup_{f \in C_{\beta, \infty}^\psi} \left\| \psi(\delta) \int_{-\infty}^{+\infty} f_\beta^\psi \left( x + \frac{t}{\delta} \right) (\hat{\varphi}(t) + \hat{\nu}(t)) dt \right\|_C \\ &= \sup_{f \in C_{\beta, \infty}^\psi} \left\| \psi(\delta) \int_{-\infty}^{+\infty} f_\beta^\psi \left( x + \frac{t}{\delta} \right) \hat{\varphi}(t) dt \right\|_C + O(\psi(\delta)A(\nu)). \end{aligned} \tag{59}$$

Repeating the arguments in [2, p. 12], one can easily show that the Fourier series of the function

$$f_\varphi(x) = \int_{-\infty}^{+\infty} f_\beta^\psi \left( x + \frac{t}{\delta} \right) \hat{\varphi}(t) dt$$

has the form

$$S[f_\varphi] = \sum_{k=1}^{\infty} \left( \frac{k^2}{2\delta^2} + \frac{k}{\delta^2} \right) \frac{1}{\psi(\delta)} (a_k \cos kx + b_k \sin kx),$$

where  $a_k$  and  $b_k$  are the Fourier coefficients of the function  $f$ . Thus,

$$\int_{-\infty}^{+\infty} f_\beta^\psi \left( x + \frac{t}{\delta} \right) \hat{\varphi}(t) dt = \frac{1}{\delta^2 \psi(\delta)} \left( \frac{f_0^{(2)}(x)}{2} + f_0^{(1)}(x) \right), \tag{60}$$

where  $f_0^{(1)}(\cdot)$  and  $f_0^{(2)}(\cdot)$  are the  $(\psi, \beta)$ -derivatives of the functions  $f(\cdot)$  (in Stepanets' sense) for  $\beta = 0$  and  $\psi(t) = \frac{1}{t}$  and  $\psi(t) = \frac{1}{t^2}$ , respectively. Combining (59) and (60), we conclude that

$$\mathcal{E} \left( C_{\beta, \infty}^\psi; B_\delta \right)_C = \frac{1}{\delta^2} \sup_{f \in C_{\beta, \infty}^\psi} \left\| \frac{f_0^{(2)}(x)}{2} + f_0^{(1)}(x) \right\|_C + O(\psi(\delta)A(\nu)), \quad \delta \rightarrow \infty. \tag{61}$$

Inequalities (2.14) and (2.15) from [13] and relations (46), (47), (54), (56), and (58) imply the following estimate of the integral  $A(\nu)$ :

$$A(\nu) = O \left( \frac{1}{\delta^3 \psi(\delta)} \int_1^\delta u^2 \psi(u) du + \frac{1}{\delta^2 \psi(\delta)} \int_\delta^\infty u \psi(u) du \right), \quad \delta \rightarrow \infty.$$

This and (61) yield (9).

Theorem 1 is proved.



Note that Theorem 1 holds, e.g., for functions  $\psi \in \mathfrak{M}$  of the form (for  $t \geq 1$ ):  $\psi(t) = \frac{1}{t^2} \ln^\alpha(t + K)$ ,  $K > 0$ ,  $\alpha < -1$ ;  $\psi(t) = \frac{1}{t^r} \ln^\alpha(t + K)$ ,  $\psi(t) = \frac{1}{t^r} \arctan t$ , and  $\psi(t) = \frac{1}{t^r}(K + e^{-t})$ ,  $r > 2$ ,  $K > 0$ ,  $\alpha \in \mathbb{R}$ .

Further, we find the solution of the Kolmogorov–Nikol’skii problem for biharmonic Poisson integrals and the classes  $C_{\beta, \infty}^\psi$  of continuous functions in the case where  $\psi \in \mathfrak{M}$ ; in particular, for the classes containing infinitely differentiable functions.

**Theorem 2.** *If  $\psi$  belongs to  $\mathfrak{M}$ , the function  $g(u) = u^2\psi(u)$  is convex downward for  $u \in [b, \infty)$ ,  $b \geq 1$ , and*

$$\int_1^\infty ug(u)du < \infty, \tag{62}$$

then the following asymptotic equality is true as  $\delta \rightarrow \infty$ :

$$\mathcal{E}\left(C_{\beta, \infty}^\psi; B_\delta\right)_C = \frac{1}{\delta^2} \sup_{f \in C_{\beta, \infty}^\psi} \left\| \frac{f_0^{(2)}(x)}{2} + f_0^{(1)}(x) \right\|_C + O\left(\frac{1}{\delta^3}\right), \tag{63}$$

where  $f_0^{(1)}(\cdot)$  and  $f_0^{(2)}(\cdot)$  are the  $(\psi, \beta)$ -derivatives of the function  $f(\cdot)$  for  $\beta = 0$  and  $\psi(t) = \frac{1}{t}$  and  $\psi(t) = \frac{1}{t^2}$ , respectively,

**Proof.** Let  $\tau(u) = \varphi(u) + v(u)$ , where  $\varphi(u)$  and  $v(u)$  are the functions defined by relations (10) and (11). We prove the summability of transforms  $\hat{\varphi}_\delta(t)$  and  $\hat{\mu}_\delta(t)$  of the form (12) and (13) on the entire real axis. First, we show that the integral  $A(\varphi)$  of the form (14) is convergent. To this end, we split the set  $(-\infty, \infty)$  into two subsets  $(-\infty, \delta) \cup (\delta, +\infty)$  and  $[-\delta, \delta]$ .

We estimate the integral  $A(\varphi)$  for  $|t| > \delta$ . Consider the integral

$$\int_0^\infty \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du$$

in each interval  $[0; 1/\delta)$  and  $[1/\delta; \infty)$ . Thus, we get

$$\int_0^\infty \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du = \left( \int_0^{1/\delta} + \int_{1/\delta}^\infty \right) \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du. \tag{64}$$

As follows from (10), for  $u \in \left[0, \frac{1}{\delta}\right)$ , we have

$$\varphi(0) = 0, \quad \varphi\left(\frac{1}{\delta}\right) = \frac{3\psi(1)}{2\delta^2\psi(\delta)}, \quad \varphi'(0) = \frac{\psi(1)}{\delta\psi(\delta)}, \quad \text{and} \quad \varphi'\left(\frac{1}{\delta} - 0\right) = \frac{2\psi(1)}{\delta\psi(\delta)}.$$

Integrating the first integral on the right-hand side of equality (64) by parts twice, we obtain

$$\int_0^{1/\delta} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du = \frac{3\psi(1)}{2t\delta^2\psi(\delta)} \sin\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) + \frac{2\psi(1)}{t^2\delta\psi(\delta)} \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \int_0^{1/\delta} \varphi''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du. \tag{65}$$

Note that, in view of the convexity of the function  $g(u)$  and condition (62), relations (20) and (21) are true. Thus, in view of the facts that  $\lim_{u \rightarrow \infty} \varphi(u) = 0$  and  $\lim_{u \rightarrow \infty} \varphi'(u) = 0$ , we conclude that

$$\int_{1/\delta}^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du = -\frac{3\psi(1)}{2t\delta^2\psi(\delta)} \sin\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \frac{4\psi(1) + 3\psi'(1)}{2\delta\psi(\delta)} \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \int_{1/\delta}^{\infty} \varphi''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du. \tag{66}$$

Combining relations (64)–(66), we can write

$$\int_0^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du = -\frac{3\psi'(1)}{2t^2\delta\psi(\delta)} \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \int_0^{1/\delta} \varphi''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du - \frac{1}{t^2} \int_{1/\delta}^{\infty} \varphi''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du.$$

Thus,

$$\left| \int_0^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| \leq \frac{K_1}{t^2\delta\psi(\delta)} + \frac{1}{t^2} \int_0^{1/\delta} |\varphi''(u)| du + \frac{1}{t^2} \int_{1/\delta}^{\infty} |\varphi''(u)| du. \tag{67}$$

The function  $\varphi(\cdot)$  of the form (10) satisfies the following evident estimate on the segment  $[0, 1/\delta]$  :

$$\int_0^{1/\delta} |\varphi''(u)| du = \frac{\psi(1)}{\delta\psi(\delta)}. \tag{68}$$

Further, by using relation (17) and the fact that the function  $\psi(\delta u)$ ,  $u \in \left[\frac{1}{\delta}, \infty\right)$ , is decreasing and convex downward, we get

$$\int_{1/\delta}^{b/\delta} |\varphi''(u)| du \leq \frac{1}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \psi(\delta u) du + \frac{2\delta}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \left(u + \frac{1}{\delta}\right) |\psi'(\delta u)| du + \frac{\delta^2}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \left(\frac{u^2}{2} + \frac{u}{\delta}\right) \psi''(\delta u) du. \tag{69}$$

It is easy to see that

$$\frac{\delta^2}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \left(\frac{u^2}{2} + \frac{u}{\delta}\right) \psi''(\delta u) du = \frac{K_2}{\delta\psi(\delta)} - \frac{\delta}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \left(u + \frac{1}{\delta}\right) \psi'(\delta u) du.$$

Combining the last relation with inequality (69), by using the inequality

$$\frac{1}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \psi(\delta u) du \leq \frac{(b-1)\psi(1)}{\delta\psi(\delta)},$$

we conclude that

$$\int_{1/\delta}^{b/\delta} |\varphi''(u)| du \leq \frac{K_2}{\delta\psi(\delta)} + \frac{3\delta}{\psi(\delta)} \int_{1/\delta}^{b/\delta} \left(u + \frac{1}{\delta}\right) |\psi'(\delta u)| du \leq \frac{K_3}{\delta\psi(\delta)}. \tag{70}$$

By using relation (17) once again, in view of the facts that the function  $\psi(u)$  is decreasing on  $[1, \infty)$  and  $\lim_{u \rightarrow \infty} \psi(u) = 0$  and relations (20) and (21), we arrive at the estimate

$$\frac{1}{t^2} \int_{1/\delta}^{\infty} |\varphi''(u)| du \leq \frac{K_4}{t^2\delta\psi(\delta)}.$$

This and relations (67)–(70) imply that

$$\left| \int_0^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| \leq \frac{K}{t^2\delta\psi(\delta)}$$

and, hence,

$$\int_{|t| \geq \delta} \left| \int_0^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt \leq \frac{2K}{\delta^2\psi(\delta)}. \tag{71}$$

We now estimate the integral  $A(\varphi)$  on the segment  $[-\delta, \delta]$ . Since condition (62) is satisfied, we have

$$\begin{aligned} & \int_{-\delta}^{\delta} \left| \int_0^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt \\ & \leq 2\delta \int_0^{\infty} |\varphi(u)| du = \frac{2\delta\psi(1)}{\psi(\delta)} \int_0^{1/\delta} \left(\frac{u^2}{2} + \frac{u}{\delta}\right) du + \frac{2\delta}{\psi(\delta)} \int_{1/\delta}^{\infty} \left(\frac{u^2}{2} + \frac{u}{\delta}\right) \psi(\delta u) du \leq \frac{K_1}{\delta^2\psi(\delta)}. \end{aligned} \quad (72)$$

The next estimate follows from relations (71) and (72) as  $\delta \rightarrow \infty$ :

$$A(\varphi) = O\left(\frac{1}{\delta^2\psi(\delta)}\right).$$

Hence, the transform  $\hat{\varphi}(t)$  (12) is summable on the entire real axis.

Further, we prove the convergence of the integral  $A(\nu)$  (34), where  $\hat{\nu}(t)$  is the Fourier transform of the function  $\nu(\cdot)$  defined by relation (11). To this end, we split the set  $(-\infty, \infty)$  into two parts  $[-\delta, \delta]$  and  $|t| > \delta$  so that

$$A(\nu) = \frac{1}{\pi} \int_{-\delta}^{\delta} \left| \int_0^{\infty} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt + \frac{1}{\pi} \int_{|t|>\delta} \left| \int_0^{\infty} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt := I_1 + I_2. \quad (73)$$

We now estimate the integral

$$I_1 = \frac{1}{\pi} \int_{-\delta}^{\delta} \left| \int_0^{\infty} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt.$$

Thus, we get

$$I_1 \leq \frac{1}{\pi} \int_{-\delta}^{\delta} \left| \int_0^{1/\delta} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt + \frac{1}{\pi} \int_{-\delta}^{\delta} \left| \int_{1/\delta}^{\infty} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt. \quad (74)$$

As already indicated, according to (11) and (37), we have  $\nu(u) = \bar{\nu} \frac{\psi(1)}{\psi(\delta)}$  for  $u \in [0, 1/\delta]$ . Hence, by using the first inequality in (40), we find

$$\begin{aligned} & \frac{1}{\pi} \int_{-\delta}^{\delta} \left| \int_0^{1/\delta} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt \leq \frac{1}{\pi} \int_{-\delta}^{\delta} \int_0^{1/\delta} |\nu(u)| du dt = \frac{\psi(1)}{\pi\psi(\delta)} \int_{-\delta}^{\delta} \int_0^{1/\delta} |\bar{\nu}(u)| du dt \\ & \leq \frac{2\delta\psi(1)}{\pi\psi(\delta)} \int_0^{1/\delta} \left(\frac{2u}{3\delta^2} + \frac{u^2}{\delta} + \frac{u^3}{2}\right) du = \frac{K}{\delta^3\psi(\delta)}. \end{aligned} \quad (75)$$

In view of condition (62) and inequality (40), we obtain the following estimate for the second integral on the right-hand side of (74):

$$\begin{aligned} & \frac{1}{\pi} \int_{-\delta}^{\delta} \left| \int_{1/\delta}^{\infty} v(u) \cos \left( ut + \frac{\beta\pi}{2} \right) du \right| dt \\ & \leq \frac{1}{\pi} \int_{-\delta}^{\delta} \int_{1/\delta}^{\infty} |v(u)| du dt \\ & = \frac{4}{3\pi\delta^3\psi(\delta)} \int_1^{\infty} u\psi(u) du + \frac{2}{\pi\delta^3\psi(\delta)} \int_1^{\infty} u^2\psi(u) du + \frac{1}{\pi\delta^3\psi(\delta)} \int_1^{\infty} u^3\psi(u) du \\ & = O \left( \frac{1}{\delta^3\psi(\delta)} \right). \end{aligned} \tag{76}$$

Relations (74)–(76) imply that

$$I_1 = O \left( \frac{1}{\delta^3\psi(\delta)} \right), \quad \delta \rightarrow \infty. \tag{77}$$

We now estimate the integral

$$I_2 = \frac{1}{\pi} \int_{|t|>\delta} \left| \int_0^{\infty} v(u) \cos \left( ut + \frac{\beta\pi}{2} \right) du \right| dt.$$

Integrating this integral by parts twice and taking into account the fact that  $v(0) = 0$  and  $v'(0) = 0$ , we conclude that

$$\begin{aligned} & \int_0^{1/\delta} v(u) \cos \left( ut + \frac{\beta\pi}{2} \right) du \\ & = \frac{1}{t} v \left( \frac{1}{\delta} \right) \sin \left( \frac{t}{\delta} + \frac{\beta\pi}{2} \right) + \frac{1}{t^2} v' \left( \frac{1}{\delta} - 0 \right) \cos \left( \frac{t}{\delta} + \frac{\beta\pi}{2} \right) - \frac{1}{t^2} \int_0^{1/\delta} v''(u) \cos \left( ut + \frac{\beta\pi}{2} \right) du. \end{aligned} \tag{78}$$

In view of (20) and (21), we get  $\lim_{u \rightarrow \infty} v(u) = 0$  and  $\lim_{u \rightarrow \infty} v'(u) = 0$ . Thus,

$$\int_{1/\delta}^{\infty} v(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du = -\frac{1}{t} v\left(\frac{1}{\delta}\right) \sin\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} v'\left(\frac{1}{\delta}\right) \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \int_{1/\delta}^{\infty} v''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du. \tag{79}$$

Combining (78) with (79), we obtain

$$\int_0^{\infty} v(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du = \frac{1}{t^2} \left( v'\left(\frac{1}{\delta} - 0\right) - v'\left(\frac{1}{\delta}\right) \right) \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \int_0^{1/\delta} v''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du - \frac{1}{t^2} \int_{1/\delta}^{\infty} v''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du.$$

According to (11) and (37), we find

$$v'\left(\frac{1}{\delta} - 0\right) = \bar{v}'\left(\frac{1}{\delta}\right) \frac{\psi(1)}{\psi(\delta)}, \tag{80}$$

$$v'\left(\frac{1}{\delta}\right) = \bar{v}'\left(\frac{1}{\delta}\right) \frac{\psi(1)}{\psi(\delta)} + \bar{v}\left(\frac{1}{\delta}\right) \frac{\delta\psi'(1)}{\psi(\delta)}. \tag{81}$$

Therefore,

$$\int_0^{\infty} v(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du = -\frac{1}{t^2} \bar{v}\left(\frac{1}{\delta}\right) \frac{\delta\psi'(1)}{\psi(\delta)} \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \left[ \int_0^{1/\delta} + \int_{1/\delta}^{\infty} \right] v''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du,$$

whence, in view of the first inequality in (40), we get

$$\left| \int_0^{\infty} v(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| \leq \frac{1}{t^2} \left( \frac{K_1}{\delta^2\psi(\delta)} + \int_0^{1/\delta} |v''(u)| du + \int_{1/\delta}^{\infty} |v''(u)| du \right). \tag{82}$$

Further, by using relations (39) and (80), the fact that  $\mu'(0) = 0$ , and the first inequality in (40), we obtain

$$\int_0^{1/\delta} |v''(u)| du = -v'\left(\frac{1}{\delta} - 0\right) = \left| \bar{v}'\left(\frac{1}{\delta}\right) \right| \frac{\psi(1)}{\psi(\delta)} \leq \frac{K_2}{\delta^2\psi(\delta)}. \tag{83}$$

Consider the second integral on the right-hand side of inequality (82) on each segment  $\left[\frac{1}{\delta}, \frac{b}{\delta}\right]$  and  $\left[\frac{b}{\delta}, \infty\right)$ . By using (42) and reasoning by analogy with the proof of relation (43), we obtain

$$\int_{1/\delta}^{b/\delta} |v''(u)| du \leq \frac{K_3}{\delta^2 \psi(\delta)}. \tag{84}$$

In view of (44) and (81), the fact that  $\lim_{u \rightarrow \infty} v'(u) = 0$ , and inequalities (40), we find

$$\int_{b/\delta}^{\infty} |v''(u)| du = - \int_{b/\delta}^{\infty} dv'(u) = \bar{v}'\left(\frac{b}{\delta}\right) \frac{\psi(b)}{\psi(\delta)} + \left| \bar{v}'\left(\frac{b}{\delta}\right) \right| \frac{|\delta \psi'(b)|}{\psi(\delta)} \leq \frac{K_4}{\delta^2 \psi(\delta)}. \tag{85}$$

It follows from (82)–(85) that

$$\left| \int_0^{\infty} v(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| \leq \frac{K}{t^2 \delta^2 \psi(\delta)}.$$

Hence, we get

$$I_2 = \frac{1}{\pi} \int_{|t| \geq \delta} \left| \int_0^{\infty} v(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt = O\left(\frac{1}{\delta^3 \psi(\delta)}\right) \text{ as } \delta \rightarrow \infty. \tag{86}$$

Combining relations (73), (77), and (86), we arrive at the following estimate for integral (34):

$$A(v) = O\left(\frac{1}{\delta^3 \psi(\delta)}\right), \quad \delta \rightarrow \infty. \tag{87}$$

Since the Fourier transforms  $\hat{\varphi}(t)$  and  $\hat{v}_\delta(t)$  are summable on the entire real axis, we conclude that relation (61) is true under the conditions of Theorem 2. Finally, by using (61) and (87), we obtain (63).

Theorem 2 is proved.

Note that conditions of Theorem 2 are satisfied by the functions  $\psi \in \mathfrak{M}$  which have have the following form for  $t \geq 1$  :

$$\psi(t) = \frac{\ln^\alpha(t + K)}{t^r}, \quad \psi(t) = \frac{1}{t^r}(K + e^{-t}), \quad \text{where } r > 4, K > 0, \alpha \in \mathbb{R};$$

$$\psi(t) = t^r e^{-Kt^\alpha}, \quad \psi(t) = \ln^r(t + e)e^{-Kt^\alpha}, \quad K > 0, \alpha > 0, r \in \mathbb{R}.$$

Assume that the function  $\mu(\cdot) = \mu(\psi; \cdot)$  is connected with the function  $\psi \in \mathfrak{M}$  by relations (1). Theorem 2 yields the following corollary:

**Corollary 1.** If  $\psi$  belongs to  $\mathfrak{M}_\infty$ , the function  $g(u)$  is convex downward for  $u \in [b, \infty)$ ,  $b \geq 1$ , and

$$\lim_{t \rightarrow \infty} \mu(\psi; t) = \infty, \quad (88)$$

then the asymptotic equality (63) is true as  $\delta \rightarrow \infty$ .

**Proof.** We check that condition (88) guarantees the convergence of the integral  $\int_1^\infty ug(u)du$ , i.e., the validity of (62). As follows from [4, p. 164] [see relation (12.24)], for any  $\psi \in \mathfrak{M}$ , the following inequality is true:

$$\frac{\psi(t)}{|\psi'(t)|} \leq 2(\eta(t) - t) \quad \forall t \geq 1. \quad (89)$$

In view of (89), for any  $r \geq 0$ , we get

$$(t^r \psi(t))' = rt^{r-1} \psi(t) - t^r |\psi'(t)| \leq t^r |\psi'(t)| \left( 2r \frac{\eta(t) - t}{t} - 1 \right). \quad (90)$$

By virtue of (88), the ratio  $(\eta(t) - t)/t$  approaches zero as  $t \rightarrow \infty$ . Thus, by using (90), we conclude that, for any  $r \geq 0$ , there exists a number  $t_0 = t_0(r, \psi)$  such that the function  $t^r \psi(t)$  does not increase for  $t > t_0$ . Then

$$\int_1^\infty ug(u)du = \int_1^\infty \frac{u^5 \psi(u)}{u^2} du \leq K \int_1^\infty \frac{du}{u^2} < \infty.$$

Note that, under the conditions of Theorems 1 and 2, equalities (9) and (63) give the solution of the Kolmogorov–Nicol'skii problem for the classes  $C_{\beta, \infty}^\psi$  and biharmonic Poisson integrals in the uniform metric in the case where the functions  $\psi(t)$  decrease to zero as  $t \rightarrow \infty$  faster than the function  $\frac{1}{t^2}$ , which specifies the order of saturation of the method of linear approximation generated by the operator  $B_\delta$ .

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