Ukrainian Mathematical Journal, Vol. 61, No. 12, 2009

APPROXIMATION OF FUNCTIONS FROM THE CLASS $C^{\psi}_{\beta,\infty}$ BY POISSON INTEGRALS IN THE UNIFORM METRIC

T. V. Zhyhallo and Yu. I. Kharkevych

UDC 517.5

We obtain asymptotic equalities for upper bounds of deviations of the Poisson integrals on the class of continuous functions $C^{\psi}_{\beta,\infty}$ in the metric of the space C.

1. Statement of the Problem and Auxiliary Assertions

Let $f(\cdot)$ be a 2π-periodic Lebesgue-summable function $(f \in L_1)$. The Poisson integral of the function f is introduced (see [1, p. 154] or [2, p. 161]) as the function $P(\rho; f; x)$ defined by the equality

$$P(\rho; f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left\{\frac{1}{2} + \sum_{k=1}^{\infty}\rho^k \cos kt\right\}dt, \quad 0 \le \rho < 1.$$

Setting $\rho = e^{-1/\delta}$, we represent the Poisson integral in the form

$$P_\delta(f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-k/\delta}\cos kt\right\}dt, \quad \delta > 0.$$

In the present paper, we consider the class $C^{\psi}_{\beta,\infty}$ introduced by Stepanets (see, e.g., [3–6])), which is defined as follows: Assume that a function f belongs to L_1 and its Fourier series has the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

Let $\psi(k)$ be an arbitrary function of a natural argument and let β be a fixed real number. If the series

$$\sum_{k=1}^{\infty}\frac{1}{\psi(k)}\left(a_k \cos\left(kx + \frac{\pi\beta}{2}\right) + b_k \sin\left(kx + \frac{\pi\beta}{2}\right)\right)$$

is the Fourier series of a certain function $\varphi \in L_1$, then φ is called the (ψ, β)-derivative of the function f and is denoted by $f^{\psi}_{\beta}(\cdot)$. Let L^{ψ}_{β} denote the subset of all functions $f \in L_1$ that have (ψ, β)-derivatives. If f belongs

Volyn National University, Luts'k, Ukraine.

Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 61, No. 12, pp. 1612–1629, December, 2009. Original article submitted July 6, 2009.

0041–5995/09/6112–1893

to L^{ψ}_{β} and f^{ψ}_{β} belongs to $\mathfrak{N}$, $\mathfrak{N} \subseteq L_1$, then one says that f belongs to $L^{\psi}_{\beta}\mathfrak{N}$. The subsets of continuous functions from L^{ψ}_{β} and $L^{\psi}_{\beta}\mathfrak{N}$ are denoted by C^{ψ}_{β} and $C^{\psi}_{\beta}\mathfrak{N}$, respectively. Further, if $\mathfrak{N}$ coincides with the unit ball of the space L_∞, i.e.,

$$\mathfrak{N} = \left\{ f^{\psi}_{\beta} \in L_\infty \colon \operatorname*{ess\,sup}_{t} \left| f^{\psi}_{\beta}(t) \right| \le 1 \right\},$$

then the classes $C^{\psi}_{\beta}\mathfrak{N}$ are denoted by $C^{\psi}_{\beta,\infty}$.

In the present paper, we study the asymptotic behavior of the quantity

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_\delta\right)_C = \sup_{f \in C^{\psi}_{\beta,\infty}} \left\| f(\cdot) - P_\delta(f;\cdot) \right\|_C \tag{1}$$

as $\delta \to \infty$.

Following Stepanets [6, p. 198], we call the problem of finding asymptotic relations for quantity (1) as $\delta \to \infty$ the *Kolmogorov–Nikol'skii problem* for Poisson integrals on the class $C^{\psi}_{\beta,\infty}$ in the uniform metric.

Let $\mathfrak{M}$ denote the set of functions $\psi(\cdot)$ that satisfy the conditions

$$\mathfrak{M} = \left\{ \psi(t) \colon \ \psi(t) > 0, \ \ \psi(t_1) - 2\psi\left((t_1 + t_2)/2\right) + \psi(t_2) \ge 0 \ \ \forall t_1, t_2 \in [1, \infty), \ \ \lim_{t \to \infty} \psi(t) = 0 \right\}.$$

Let $\mathfrak{M}'$ denote the set of functions $\psi \in \mathfrak{M}$ for which

$$\int_1^\infty \frac{\psi(t)}{t} dt < \infty.$$

Using the characteristics

$$\eta(t) = \eta(\psi; t) = \psi^{-1} \frac{\psi(t)}{2}, \qquad \mu(t) = \mu(\psi; t) = \frac{t}{\eta(t) - t}, \tag{2}$$

where ψ^{-1} is the function inverse to ψ, one customarily considers (see, e.g., [5, p. 93] or [6, p. 160]) the following subsets of the set $\mathfrak{M}$:

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M} \colon \ 0 < \mu(\psi; t) \le K \ \ \forall t \ge 1\},$$

$$\mathfrak{M}_C = \{\psi \in \mathfrak{M} \colon \ 0 < K_1 < \mu(\psi; t) \le K_2 \ \ \forall t \ge 1\},$$

$$\mathfrak{M}_\infty = \{\psi \in \mathfrak{M} \colon \ 0 < K \le \mu(\psi; t) < \infty \ \ \forall t \ge 1\}.$$

Here and in what follows, K and K_i denote constants, generally speaking, different in different relations and dependent on ψ.

Note that, for functions $\psi \in \mathfrak{M}_0'$ $(\mathfrak{M}_0' = \mathfrak{M}_0 \cap \mathfrak{M}')$ slowly decreasing to zero, i.e., for functions ψ such that

$$\int_1^{\infty} \psi(t)dt = \infty,$$

the Kolmogorov–Nikol'skii problem was solved in [7]. The aim of the present paper is to find asymptotic equalities for upper bounds of deviations of Poisson integrals on the classes $C^{\psi}_{\beta,\infty}$ for $\beta \in R$ in the cases where $\psi \in \mathfrak{M}_C$ and $\psi \in \mathfrak{M}_\infty$, i.e., for functions $\psi(t)$ that decrease to zero as $t \to \infty$ faster than the function $1/t$, which determines the order of saturation of the linear approximation method generated by the operator P_δ.

If the Fourier transform

$$\hat{\tau}(t) = \hat{\tau}_\delta(t) = \frac{1}{\pi}\int_0^{\infty} \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du \tag{3}$$

of the function $\tau(\cdot)$ defined by the equalities

$$\tau(u) = \tau_\delta(u;\psi) = \begin{cases} \left(1 - e^{-u}\right)\dfrac{\psi(1)}{\psi(\delta)}, & 0 \le u \le \dfrac{1}{\delta}, \\ \left(1 - e^{-u}\right)\dfrac{\psi(\delta u)}{\psi(\delta)}, & u \ge \dfrac{1}{\delta} \end{cases} \tag{4}$$

is summable on the entire number axis, i.e., the integral $A(\tau)$

$$A(\tau) = \int_{-\infty}^{\infty} \left|\hat{\tau}_\delta(t)\right|dt \tag{5}$$

is convergent, then, for any $f \in C^{\psi}_{\beta,\infty}$, the following equality holds at every point $x \in R$:

$$f(x) - P_\delta(f;x) = \psi(\delta)\int_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right)\hat{\tau}_\delta(t)dt, \quad \delta > 0. \tag{6}$$

Note that, relation (6) can be obtained by repeating the arguments used in [6, p. 183]. Thus, to find asymptotic equalities for quantity (1) as $\delta \to \infty$ in the case where $\psi \in \mathfrak{M}_C$, $\psi \in \mathfrak{M}_\infty$, and $\beta \in R$, it is necessary to find conditions under which the Fourier transform $\hat{\tau}(t)$ is summable on the entire number axis.

2. Asymptotic Equalities for Upper Bounds of Deviations of Poisson Integrals from Functions of the Class $C^{\psi}_{\beta,\infty}$ in the Uniform Metric

The following statement is true:

Theorem 1. *Suppose that* $\psi \in \mathfrak{M}_C$, *the function* $g(u) = u\psi(u)$ *is convex downward on* $\left[b, \infty\right)$, $b \ge 1$, *and*

$$\int\limits_{1}^{\infty} \psi(u)du < \infty. \tag{7}$$

Then the following asymptotic equality holds as $\delta \to \infty$:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_\delta\right)_C = \frac{1}{\delta} \sup_{f \in C^{\psi}_{\beta,\infty}} \left\| f_0^{(1)}(x) \right\|_C + O\left(\frac{1}{\delta^2} \int\limits_{1}^{\delta} t\psi(t)dt + \frac{1}{\delta} \int\limits_{\delta}^{\infty} \psi(t)dt \right), \tag{8}$$

where $f_0^{(1)}$ *is the* (ψ, β)*-derivative of the function* f *for* $\psi(t) = 1/t$ *and* $\beta = 0.$

Prior to the proof of Theorem 1, we consider the following lemma:

Lemma 1. *Suppose that all conditions of Theorem 1 are satisfied. Then a Fourier transform* $\hat{\tau}(t)$ *of the form (3) for the function* $\tau(u)$ *defined by (4) is summable on the entire number axis, i.e., integral (5) is convergent.*

Proof of Lemma 1. We set $\tau(u) = \varphi(u) + \nu(u),$ where

$$\varphi(u) = \begin{cases} u\dfrac{\psi(1)}{\psi(\delta)}, & 0 \le u < \dfrac{1}{\delta}, \\[2ex] u\dfrac{\psi(\delta u)}{\psi(\delta)}, & u \ge \dfrac{1}{\delta}, \end{cases} \tag{9}$$

$$\nu(u) = \begin{cases} (1 - e^{-u} - u)\dfrac{\psi(1)}{\psi(\delta)}, & 0 \le u \le \dfrac{1}{\delta}, \\[2ex] (1 - e^{-u} - u)\dfrac{\psi(\delta u)}{\psi(\delta)}, & u \ge \dfrac{1}{\delta}, \end{cases} \tag{10}$$

and verify that the Fourier transforms

$$\hat{\varphi}(t) = \hat{\varphi}_\delta(t) = \frac{1}{\pi} \int\limits_{0}^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du, \tag{11}$$

$$\hat{\nu}(t) = \hat{\nu}_\delta(t) = \frac{1}{\pi} \int\limits_{0}^{\infty} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \tag{12}$$

of the functions φ and $\nu,$ respectively, are summable on the entire number axis. Thus, it is necessary to show that the following integrals are convergent:

$$A(\varphi) = \int\limits_{-\infty}^{\infty} \left|\hat{\varphi}_\delta(t)\right| dt, \tag{13}$$

$$A(\nu) = \int\limits_{-\infty}^{\infty} \left|\hat{\nu}_{\delta}(t)\right| dt. \tag{14}$$

First, we prove the convergence of integral (13). According to Theorem 1 in [8], for the convergence of the integral $A(\varphi)$ it is necessary and sufficient that the following integrals be convergent:

$$\int\limits_{0}^{1/2} u|d\varphi'(u)|, \qquad \int\limits_{1/2}^{\infty} |u-1||d\varphi'(u)|,$$

$$\left|\sin\frac{\beta\pi}{2}\right| \int\limits_{0}^{\infty} \frac{|\varphi(u)|}{u} du, \qquad \int\limits_{0}^{1} \frac{|\varphi(1-u)-\varphi(1+u)|}{u} du.$$

In follows from (9) that

$$\varphi''(u) = 0, \quad u \in \left[0, \frac{1}{\delta}\right),$$

and

$$\psi(\delta)\left|d\varphi' u)\right| \le \big(2\delta|\psi'(\delta u)| + u\delta^2\psi''(\delta u)\big)du, \quad \psi \in \mathfrak{M}, \quad \text{for} \quad u \ge \frac{1}{\delta}. \tag{15}$$

Since

$$\int\limits_{0}^{1/2} u|d\varphi'(u)| = \int\limits_{1/\delta}^{1/2} u|d\varphi'(u)| \le \int\limits_{1/\delta}^{\infty} u|d\varphi'(u)|$$

and

$$\int\limits_{1/2}^{\infty} |u-1||d\varphi'(u)| \le \int\limits_{1/\delta}^{\infty} u|d\varphi'(u)|,$$

we obtain an estimate for the integral

$$\int\limits_{1/\delta}^{\infty} u|d\varphi'(u)|$$

on each of the intervals $[1/\delta, b/\delta)$ and $[b/\delta, \infty)$ (for $\delta > 2b$). Taking (15) into account, we get

$$\int\limits_{1/\delta}^{b/\delta} u|d\varphi'(u)| \le \frac{2\delta}{\psi(\delta)} \int\limits_{1/\delta}^{b/\delta} u|\psi'(\delta u)|du + \frac{\delta^2}{\psi(\delta)} \int\limits_{1/\delta}^{b/\delta} u^2\psi''(\delta u)du.$$

Integrating both integrals on the right-hand side of the last inequality by parts and taking into account that $\psi(\delta u) \le \psi(1)$ for $u \in [1/\delta, b/\delta)$, we get

$$\int_{1/\delta}^{b/\delta} u\left|d\varphi'(u)\right| \le \frac{K_1}{\delta\psi(\delta)}.$$

Further, we show that the following relations are true:

$$\lim_{u\to\infty} u\psi(u) = 0, \tag{16}$$

$$\lim_{u\to\infty} u^2\psi'(u) = 0. \tag{17}$$

Since the function $g(u) = u\psi(u)$ is convex downward for $u \ge b \ge 1$, the following cases are possible: either

$$\lim_{u\to\infty} g(u) = 0,$$

or

$$\lim_{u\to\infty} g(u) = K > 0,$$

or

$$\lim_{u\to\infty} g(u) = \infty.$$

Let

$$\lim_{u\to\infty} g(u) = K > 0.$$

Then there exists $0 < K_1 < K$ such that, for all $u \ge 1$, one has $g(u) > K_1$ and, hence,

$$\psi(u) > \frac{K_1}{u},$$

which contradicts the fact that, according to condition (7), the function $\psi(u)$ is summable on $[1, \infty)$.

Now assume that

$$\lim_{u\to\infty} g(u) = \infty,$$

i.e., for any $M > 0$, there exists $N > 0$ such that $g(u) > M$ for all $u > N$. Then

$$\int_1^x \psi(u)du = \int_1^N \psi(u)du + \int_N^x \frac{g(u)}{u}du > K_2 + \int_N^x \frac{M}{u}du = K_2 + M(\ln x - \ln N).$$

We again arrive at a contradiction with the condition of the summability of the function $\psi(u)$ on the interval $[1, \infty)$. It follows from the results presented above that relation (16) is true.

We now prove relation (17). The function $g'(u)$ is summable on $[1,\infty)$, whence

$$\lim_{u\to\infty}\int\limits_{u/2}^{u} g'(x)dx = 0.$$

Since, the function $g(u)$ is convex downward for $u \geq b \geq 1$, we conclude that the function $(-g'(u))$ does not increase for $u \geq b$, and, hence,

$$-\int\limits_{u/2}^{u} g'(x)dx > -\Big(u-\frac{u}{2}\Big)\big(\psi(u)+u\psi'(u)\big) = -\frac{1}{2}\left(u\psi(u)+u^2\psi'(u)\right).$$

This and relation (16) yield (17).

Taking into account that the function $g(u)$, $u \geq b \geq 1$, is convex downward and using relations (16) and (17), we obtain

$$\int\limits_{b/\delta}^{\infty} u|d\varphi'(u)| = \int\limits_{b/\delta}^{\infty} u d\varphi'(u) = \lim_{u\to\infty} u\varphi'(u) - \frac{b}{\delta}\varphi'\left(\frac{b}{\delta}\right) + \varphi\left(\frac{b}{\delta}\right) = \frac{K}{\delta\psi(\delta)}.$$

Thus,

$$\int\limits_{0}^{1/2} u|d\varphi'(u)| = O\left(\frac{1}{\delta\psi(\delta)}\right) \quad \text{and} \quad \int\limits_{1/2}^{\infty} |u-1||d\varphi'(u)| = O\left(\frac{1}{\delta\psi(\delta)}\right) \quad \text{as} \quad \delta\to\infty. \tag{18}$$

Further, taking into account relation (9) and the inequality

$$\int\limits_{1}^{\infty}\psi(u)du \leq K,$$

we get

$$\int\limits_{0}^{\infty}\frac{|\varphi(u)|}{u}du = \int\limits_{0}^{\infty}\frac{\varphi(u)}{u}du = \frac{\psi(1)}{\delta\psi(\delta)} + \frac{1}{\delta\psi(\delta)}\int\limits_{1}^{\infty}\psi(u)du = O\left(\frac{1}{\delta\psi(\delta)}\right).$$

Finally, we estimate the integral

$$\int\limits_{0}^{1}|\varphi(1-u)-\varphi(1+u)|\,\frac{du}{u}.$$

For this purpose, we represent this integral as a sum of two integrals:

$$\int\limits_{0}^{1}\frac{|\varphi(1-u)-\varphi(1+u)|}{u}du = \int\limits_{0}^{1-1/\delta}\frac{|\varphi(1-u)-\varphi(1+u)|}{u}du + \int\limits_{1-1/\delta}^{1}\frac{|\varphi(1-u)-\varphi(1+u)|}{u}du. \tag{19}$$

We estimate the first term on the right-hand side of (19) by adding and subtracting the quantity $(-2u)$ under the modulus sign in the integrand. As a result, we get

$$\int_0^{1-1/\delta} \frac{|\varphi(1-u)-\varphi(1+u)|}{u}du = \int_0^{1-1/\delta} \frac{|\varphi(1-u)-\varphi(1+u)-2u|}{u}du + O(1). \tag{20}$$

It follows from (9) that, for $u \in [0, 1-1/\delta]$, we have

$$1-u = 1-\frac{\psi(\delta)}{\psi(\delta(1-u))}\varphi(1-u), \qquad 1+u = 1-\frac{\psi(\delta)}{\psi(\delta(1+u))}\varphi(1+u).$$

Then

$$\int_0^{1-1/\delta} \frac{|\varphi(1-u)-\varphi(1+u)-2u|}{u}du$$

$$\leq \int_0^{1-1/\delta} |\varphi(1-u)| \left|1-\frac{\psi(\delta)}{\psi(\delta(1-u))}\right| \frac{du}{u} + \int_0^{1-1/\delta} |\varphi(1+u)| \left|1-\frac{\psi(\delta)}{\psi(\delta(1+u))}\right| \frac{du}{u}.$$

Since the function $\varphi(\cdot)$ satisfies the conditions of Lemma 2 in [8], we have

$$|\varphi(u)| \leq |\varphi(0)| + |\varphi(1)| + \int_0^{1/2} u\left|d\varphi'(u)\right| + \int_{1/2}^{\infty} |u-1|\left|d\varphi'(u)\right| := H(\varphi).$$

Thus,

$$\int_0^{1-1/\delta} \frac{|\varphi(1-u)-\varphi(1+u)-2u|}{u}du$$

$$= H(\varphi) O\left(\int_0^{1-1/\delta} \frac{|\psi(\delta(1-u))-\psi(\delta)|}{u\psi(\delta(1-u))}du + \int_0^{1-1/\delta} \frac{|\psi(\delta(1+u))-\psi(\delta)|}{u\psi(\delta(1+u))}du \right). \tag{21}$$

Taking into account relation (9) and estimates (18) and using (16), we get

$$H(\varphi) = O\left(\frac{1}{\delta\psi(\delta)}\right), \quad \delta \to \infty. \tag{22}$$

It was established in [7] that the following estimates hold for functions $\psi \in \mathfrak{M}_0$ as $\delta \to \infty$:

$$\int_0^{1-1/\delta} \frac{|\psi(\delta(1-u)) - \psi(\delta)|}{u\psi(\delta(1-u))} du = O(1), \qquad \int_0^{1-1/\delta} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u\psi(\delta(1+u))} du = O(1);$$

these estimates are also true for functions $\psi \in \mathfrak{M}_C$.

Combining relations (20)–(22), we get

$$\int_0^{1-1/\delta} \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = O\left(\frac{1}{\delta\psi(\delta)}\right).$$

By analogy, one can easily verify that the same estimate holds for the second term on the right-hand side of (19). Therefore,

$$\int_0^1 |\varphi(1-u) - \varphi(1+u)| \frac{du}{u} = O\left(\frac{1}{\delta\psi(\delta)}\right), \quad \delta \to \infty.$$

Thus, we have established the convergence of integral (13) in the case where $\psi \in \mathfrak{M}$, the function $g(u) = u\psi(u)$ is convex downward on $\big[b, \infty\big)$, $b \geq 1$, and condition (7) is satisfied. Let us prove the convergence of integral (14). To this end, by virtue of Theorem 1 in [8], it is necessary to estimate the integrals

$$\int_0^{1/2} u|d\nu'(u)|, \qquad \int_{1/2}^{\infty} |u-1||d\nu'(u)|, \tag{23}$$

$$\left|\sin\frac{\beta\pi}{2}\right| \int_0^{\infty} \frac{|\nu(u)|}{u} du, \qquad \int_0^1 \frac{|\nu(1-u) - \nu(1+u)|}{u} du, \tag{24}$$

where $\nu(u)$ is the function given by (10), which is defined and continuous for all $u \geq 0$.

To estimate the first integral in (23), we divide the segment $[0; 1/2]$ into the two parts $[0; 1/\delta]$ and $[1/\delta; 1/2]$. It follows from (10) that

$$\nu''(u) = -e^{-u}\frac{\psi(1)}{\psi(\delta)} \quad \text{for} \quad u \in \left[0, \frac{1}{\delta}\right).$$

Therefore,

$$\int_0^{1/\delta} u|d\nu'(u)| = \frac{\psi(1)}{\psi(\delta)} \int_0^{1/\delta} ue^{-u} du \leq \frac{\psi(1)}{\psi(\delta)} \int_0^{1/\delta} u du = O\left(\frac{1}{\delta^2\psi(\delta)}\right). \tag{25}$$

It also follows from relation (10) and properties of the function $\psi \in \mathfrak{M}$ that, for $u \geq 1/\delta$, one has

$$\left|d\nu'(u)\right| \leq \left\{\left|\overline{\nu}(u)\right| \frac{\delta^2\psi''(\delta u)}{\psi(\delta)} + 2\left|\overline{\nu}'(u)\right| \frac{\delta|\psi'(\delta u)|}{\psi(\delta)} + \left|\overline{\nu}''(u)\right| \frac{\psi(\delta u)}{\psi(\delta)}\right\} du, \tag{26}$$

where $\overline{\nu}(u) = 1 - e^{-u} - u$. Using the inequalities

$$|\overline{\nu}(u)| \le \frac{u^2}{2}, \qquad \left|\overline{\nu}'(u)\right| \le u, \qquad \left|\overline{\nu}''(u)\right| \le 1,$$

we rewrite relation (26) in the form

$$\left|d\nu'(u)\right| \le \left\{u^2\frac{\delta^2\psi''(\delta u)}{2\psi(\delta)} + 2u\frac{\delta|\psi'(\delta u)|}{\psi(\delta)} + \frac{\psi(\delta u)}{\psi(\delta)}\right\} du, \quad \psi \in \mathfrak{M}. \tag{27}$$

Using (27), we obtain the following relation for the first integral in (23) on the segment $[1/\delta; 1/2]$:

$$\int\limits_{1/\delta}^{1/2} u|d\nu(u)| \le \frac{1}{\psi(\delta)}\int\limits_{1/\delta}^{1/2} \frac{u^3}{2}\delta^2\psi''(\delta u)du + \frac{2}{\psi(\delta)}\int\limits_{1/\delta}^{1/2} u^2\delta\left|\psi'(\delta u)\right| du + \frac{1}{\psi(\delta)}\int\limits_{1/\delta}^{1/2} u\psi(\delta u)du.$$

Taking the first integral on the right-hand side of the last inequality, we obtain

$$\int\limits_{1/\delta}^{1/2} u|d\nu(u)| \le \frac{1}{\psi(\delta)}\,\frac{u^3}{2}\delta\psi'(\delta u)\Bigg|_{1/\delta}^{1/2} + \frac{7}{2\psi(\delta)}\int\limits_{1/\delta}^{1/2} u^2\delta\left|\psi'(\delta u)\right| du + \frac{1}{\psi(\delta)}\int\limits_{1/\delta}^{1/2} u\psi(\delta u)du. \tag{28}$$

Further, we use the following statements:

Proposition 1 [6, p. 161]. *A function $\psi \in \mathfrak{M}$ belongs to $\mathfrak{M}_C$ if and only if the quantity*

$$\alpha(t) = \frac{\psi(t)}{t\left|\psi'(t)\right|}, \qquad \psi'(t) = \psi'(t+0),$$

satisfies the condition $0 < K_1 \le \alpha(t) \le K_2 \;\; \forall t \ge 1$.

Proposition 2 [6, p. 175]. *A function $\psi \in \mathfrak{M}$ belongs to $\mathfrak{M}_0$ if and only if, for an arbitrary fixed number $c > 1$, there exists a constant K such that the following inequality holds for all $t \ge 1$:*

$$\frac{\psi(t)}{\psi(ct)} \le K.$$

Using the conditions of Proposition 1, for $\psi \in \mathfrak{M}_C$ we get

$$\frac{1}{\psi(\delta)}\int\limits_{1/\delta}^{1/2} u^2\delta\left|\psi'(\delta u)\right| du \le \frac{K}{\psi(\delta)}\int\limits_{1/\delta}^{1/2} u\psi(\delta u)du.$$

Then, using (28) and taking into account Proposition 2 (which is also true for functions $\psi \in \mathfrak{M}_C$) and the inequality

$$\int\limits_{1}^{\delta} u\psi(u)du \ge K,$$

we obtain

$$\int\limits_{1/\delta}^{1/2} u|d\nu(u)| \le K_1 + \frac{K_2}{\delta^2\psi(\delta)} + \frac{K_3}{\psi(\delta)}\int\limits_{1/\delta}^{1/2} u\psi(\delta u)du \le \frac{K}{\delta^2\psi(\delta)}\int\limits_{1}^{\delta} u\psi(u)du. \tag{29}$$

Combining (25) and (29), we get

$$\int\limits_{0}^{1/2} u|d\nu(u)| = O\left(\frac{1}{\delta^2\psi(\delta)}\int\limits_{1}^{\delta} u\psi(u)du\right), \quad \delta\to\infty. \tag{30}$$

Let us estimate the second integral in (23). For the function $\overline{\nu}(u) = 1 - e^{-u} - u$, we have $|\overline{\nu}(u)| \le u$, $\left|\overline{\nu}'(u)\right| \le 1$, and $\left|\overline{\nu}''(u)\right| = e^{-u}$. Taking this into account and using (26), we obtain the following relation for $\delta \ge 2$:

$$\int\limits_{1/2}^{\infty} |u-1||d\nu'(u)| \le \int\limits_{1/2}^{\infty} u|d\nu'(u)|$$

$$\le \frac{1}{\psi(\delta)}\int\limits_{1/2}^{\infty} ue^{-u}\psi(\delta u)du + \frac{2\delta}{\psi(\delta)}\int\limits_{1/2}^{\infty} u\left|\psi'(\delta u)\right|du + \frac{\delta^2}{\psi(\delta)}\int\limits_{1/2}^{\infty} u^2\psi''(\delta u)du. \tag{31}$$

Let us estimate the first integral on the right-hand side of (31). Since the function $\psi(\delta u)$, $\delta \ge 2$, decreases for $u \in [1/2, \infty]$, taking Proposition 2 into account we get

$$\frac{1}{\psi(\delta)}\int\limits_{1/2}^{\infty} ue^{-u}\psi(\delta u)du \le \frac{\psi(\delta/2)}{\psi(\delta)}\int\limits_{1/2}^{\infty} ue^{-u}du = O(1). \tag{32}$$

Integrating the third integral on the right-hand side of inequality (31) by parts and using equality (17) and Propositions 1 and 2, we obtain the following relation for the functions $\psi(\delta u) \in \mathfrak{M}_C$, $u \ge 1/2$, $\delta \ge 2$:

$$\frac{\delta^2}{\psi(\delta)}\int\limits_{1/2}^{\infty} u^2\psi''(\delta u)du = \frac{\delta}{\psi(\delta)}\int\limits_{1/2}^{\infty} u^2 d\psi'(\delta u)$$

$$= \frac{\delta}{\psi(\delta)}\lim_{u\to\infty} u^2\psi'(\delta u) + \frac{(\delta/2)|\psi'(\delta/2)|}{2\psi(\delta)} + \frac{2\delta}{\psi(\delta)}\int\limits_{1/2}^{\infty} u\left|\psi'(\delta u)\right|du$$

$$\le K_1 + \frac{2\delta}{\psi(\delta)}\int\limits_{1/2}^{\infty} u\left|\psi'(\delta u)\right|du. \tag{33}$$

It follows from (31)–(33) that

$$\int_{1/2}^{\infty} |u-1||d\nu'(u)| \le K_2 + \frac{4\delta}{\psi(\delta)} \int_{1/2}^{\infty} u\left|\psi'(\delta u)\right| du.$$

Integrating the integral on the right-hand side of the last relation again by parts and using relation (16) and Proposition 2, we obtain

$$\int_{1/2}^{\infty} |u-1||d\nu'(u)| \le K_3 + \frac{4}{\psi(\delta)} \int_{1/2}^{\infty} \psi(\delta u)du$$

$$\le K_3 + \frac{2\psi(\delta/2)}{\psi(\delta)} + \frac{4}{\psi(\delta)} \int_{1}^{\infty} \psi(\delta u)du \le K_4 + \frac{4}{\delta\psi(\delta)} \int_{\delta}^{\infty} \psi(u)du.$$

Thus, the following estimate holds as $\delta \to \infty$:

$$\int_{1/2}^{\infty} |u-1||d\nu'(u)| = O\left(1 + \frac{1}{\delta\psi(\delta)} \int_{\delta}^{\infty} \psi(u)du\right). \tag{34}$$

To estimate the first integral in (24), we divide the interval $[0;\infty)$ into the following three parts: $[0;1/\delta]$, $[1/\delta;1]$, and $[1;\infty)$. Taking into account the inequality

$$e^{-u} \le 1 - u + \frac{u^2}{2}, \quad u \ge 0, \tag{35}$$

we obtain

$$\int_{0}^{1/\delta} \frac{|\nu(u)|}{u}du = \frac{\psi(1)}{\psi(\delta)} \int_{0}^{1/\delta} \left(-1 + e^{-u} + u\right)\frac{du}{u} \le \frac{\psi(1)}{2\psi(\delta)} \int_{0}^{1/\delta} udu = O\left(\frac{1}{\delta^2\psi(\delta)}\right),$$

$$\int_{1/\delta}^{1} \frac{|\nu(u)|}{u}du \le \int_{1/\delta}^{1} u\frac{\psi(\delta u)}{2\psi(\delta)}du = O\left(\frac{1}{\delta^2\psi(\delta)} \int_{1}^{\delta} u\psi(u)du\right),$$

$$\int_{1}^{\infty} \frac{|\nu(u)|}{u}du = \frac{1}{\psi(\delta)} \int_{1}^{\infty} \psi(\delta u)\left(\frac{e^{-u}-1}{u} + 1\right)du \le \frac{1}{\psi(\delta)} \int_{1}^{\infty} \psi(\delta u)du.$$

Hence,

$$\int\limits_0^{\infty} \frac{|\nu(u)|}{u} du = O\left(\frac{1}{\delta^2 \psi(\delta)} \int\limits_1^{\delta} u\psi(u)du + \frac{1}{\delta\psi(\delta)} \int\limits_{\delta}^{\infty} \psi(u)du \right). \tag{36}$$

Let us estimate the second integral in (24). By analogy with the proof of relation (58) in [7], we obtain

$$\int\limits_0^1 |\nu(1-u) - \nu(1+u)| \frac{du}{u} = \int\limits_0^1 |\lambda(1-u) - \lambda(1+u)| \frac{du}{u} + O\left(H(\nu)\right), \tag{37}$$

where $\lambda(u) = e^{-u} + u$ and

$$H(\nu) = |\nu(0)| + |\nu(1)| + \int\limits_0^{1/2} u|d\nu'(u)| + \int\limits_{1/2}^{\infty} |u-1||d\nu'(u)|.$$

Taking into account relations (10), (30), and (34) and the inequality

$$\frac{1}{\delta^2\psi(\delta)} \int\limits_1^{\delta} u\psi(u)du \geq \frac{1}{\delta^2\psi(\delta)} \delta\psi\left(\delta\right) \int\limits_1^{\delta} du \geq K,$$

we get

$$H(\nu) = O\left(\frac{1}{\delta^2 \psi(\delta)} \int\limits_1^{\delta} u\psi(u)du + \frac{1}{\delta\psi(\delta)} \int\limits_{\delta}^{\infty} \psi(u)du \right), \quad \delta \to \infty. \tag{38}$$

Since

$$\int\limits_0^1 |\lambda(1-u) - \lambda(1+u)| \frac{du}{u} \leq K, \tag{39}$$

relations (37)–(39) yield the following estimate as $\delta \to \infty$:

$$\int\limits_0^1 |\nu(1-u) - \nu(1+u)| \frac{du}{u} = O\left(\frac{1}{\delta^2 \psi(\delta)} \int\limits_1^{\delta} u\psi(u)du + \frac{1}{\delta\psi(\delta)} \int\limits_{\delta}^{\infty} \psi(u)du \right). \tag{40}$$

Thus, according to Theorem 1 in [8], integral (14) is also convergent.

Lemma 1 is proved.

Proof of Theorem 1. Lemma 1 states that, under the conditions of Theorem 1, the Fourier transform $\hat{\tau}(t)$ (3) of the function $\tau(u) = \varphi(u) + \nu(u)$ is summable on the entire number axis. Then, for any function $f \in C^{\psi}_{\beta,\infty}$, equality (6) holds at every point $x \in R$.

Using the integral representation (6), we represent quantity (1) in the form

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_{\delta}\right)_C = \sup_{f\in C^{\psi}_{\beta,\infty}} \left\| \psi(\delta) \int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right) \hat{\tau}(t)dt \right\|_C$$

$$= \sup_{f\in C^{\psi}_{\beta,\infty}} \left\| \psi(\delta) \int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right) (\hat{\varphi}(t) + \hat{\nu}(t))\, dt \right\|_C .$$

Using (14), we obtain

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_{\delta}\right)_C = \sup_{f\in C^{\psi}_{\beta,\infty}} \left\| \psi(\delta) \int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right) \hat{\varphi}(t)dt \right\|_C + O\left(\psi(\delta)A(\nu)\right). \tag{41}$$

Repeating the arguments of [3], one can easily verify that the Fourier series of the function

$$f_{\varphi}(x) = \int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right) \hat{\varphi}(t)dt$$

has the form

$$S[f_{\varphi}] = \sum_{k=1}^{\infty} \frac{k}{\delta} \frac{1}{\psi(\delta)} \left(a_k \cos kx + b_k \sin kx\right),$$

where a_k and b_k are the Fourier coefficients of the function f. Therefore,

$$\int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right) \hat{\varphi}(t)dt = \frac{1}{\delta\psi(\delta)} f^{(1)}_0(x), \tag{42}$$

where $f^{(1)}_0(\cdot)$ is the (ψ, β)-derivative of the function $f(\cdot)$ in the Stepanets sense for $\psi(t) = 1/t$ and $\beta = 0$. Combining (41) and (42), we obtain

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_{\delta}\right)_C = \frac{1}{\delta} \sup_{f\in C^{\psi}_{\beta,\infty}} \left\| f^{(1)}_0(x) \right\|_C + O\left(\psi(\delta)A(\nu)\right), \quad \delta \to \infty. \tag{43}$$

Using inequalities (2.14) and (2.15) from [8] and relations (30), (34), (36), (38), and (40), we obtain the following estimate for the integral $A(\nu)$:

$$A(\nu) = O\left(\frac{1}{\delta^2\psi(\delta)}\int_1^{\delta} u\psi(u)du + \frac{1}{\delta\psi(\delta)}\int_{\delta}^{\infty}\psi(u)du\right), \quad \delta\to\infty.$$

This and relation (43) yield (8).

Theorem 1 is proved.

Examples of functions satisfying the conditions of Theorem 1 are functions $\psi \in \mathfrak{M}$ that have the following form for $t \geq 1$:

$$\psi(t) = \frac{1}{t}\ln^{\alpha}(t+K), \quad K>0, \quad \alpha<-1; \qquad \psi(t) = \frac{1}{t^r}\ln^{\alpha}(t+K);$$

$$\psi(t) = \frac{1}{t^r}\arctan t; \qquad \psi(t) = \frac{1}{t^r}(K+e^{-t}), \quad r>1, \quad K>0, \quad \alpha\in R.$$

In the second part of the present paper, we find a solution of the Kolmogorov–Nikol'skii problem for Poisson integrals on the classes $C^{\psi}_{\beta,\infty}$ of continuous periodic functions in the case where ψ belongs to $\mathfrak{M}_{\infty}$.

Theorem 2. *If ψ belongs to $\mathfrak{M}$, a function $g(u)$ is convex downward for $u \in [b,\infty)$, $b \geq 1$, and*

$$\int_1^{\infty} u^2\psi(u)du < \infty, \tag{44}$$

then the following asymptotic equality holds as $\delta\to\infty$:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_{\delta}\right)_C = \frac{1}{\delta}\sup_{f\in C^{\psi}_{\beta,\infty}}\left\|f_0^{(1)}(x)\right\|_C + O\left(\frac{1}{\delta^2}\right), \tag{45}$$

where $f_0^{(1)}$ is the (ψ,β)-derivative of the function f for $\psi(t)=1/t$ and $\beta=0$.

The proof of Theorem 2 is based on the following auxiliary statement:

Lemma 2. *Suppose that all conditions of Theorem 2 are satisfied. Then an integral $A(\tau)$ of the form (5) is convergent.*

Proof of Lemma 2. To establish the convergence of the integral $A(\tau)$ we represent the function $\tau(\cdot)$ (4) as the sum of the functions $\varphi(\cdot)$ and $\nu(\cdot)$ defined by (9) and (10), respectively. We investigate the convergence of integral (13). To this end, we divide the set $(-\infty,\infty)$ into the two subsets $(-\infty,\delta)\cup(\delta,+\infty)$ and $[-\delta,\delta]$.

Let us estimate the integral $A(\varphi)$ for $|t|>\delta$. To this end, we consider the integral

$$\int_0^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du$$

on each of the intervals $[0;1/\delta)$ and $[1/\delta;\infty)$:

$$\int_0^\infty \varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du = \left(\int_0^{1/\delta}+\int_{1/\delta}^\infty\right)\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du. \tag{46}$$

It follows from (9) that

$$\varphi(0)=0, \qquad \varphi\left(\frac{1}{\delta}\right)=\frac{\psi(1)}{\delta\psi(\delta)}, \qquad \text{and} \qquad \varphi'(0)=\varphi'\left(\frac{1}{\delta}-0\right)=\frac{\psi(1)}{\psi(\delta)} \quad \text{for} \quad u\in\left[0,\frac{1}{\delta}\right).$$

Integrating the first integral on the right-hand side of equality (46) twice by parts and taking into account that $\varphi''(u)=0,\ u\in[0,1/\delta),$ we obtain

$$\int_0^{1/\delta}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du=\frac{\psi(1)}{t\delta\psi(\delta)}\sin\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)+\frac{\psi(1)}{t^2\psi(\delta)}\left(\cos\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)-\cos\frac{\beta\pi}{2}\right). \tag{47}$$

By virtue of the convexity of the function $g(u)$ and condition (44), relations (16) and (17) are true. For $u\geq 1/\delta$, we get

$$\int_{1/\delta}^\infty\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du$$

$$=-\frac{\psi(1)}{t\delta\psi(\delta)}\sin\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)-\frac{1}{t^2}\left(\frac{\psi(1)}{\psi(\delta)}+\frac{\psi'(1)}{\psi(\delta)}\right)\cos\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)$$

$$-\frac{1}{t^2}\int_{1/\delta}^\infty\varphi''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du. \tag{48}$$

Combining relations (46)–(48), we obtain

$$\int_0^\infty\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du$$

$$=-\frac{1}{t^2\psi(\delta)}\left(\psi(1)\cos\frac{\beta\pi}{2}+\psi'(1)\cos\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)\right)-\frac{1}{t^2}\int_{1/\delta}^\infty\varphi''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du.$$

Thus,

$$\left|\int_0^\infty\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|\leq\frac{K}{t^2\psi(\delta)}+\frac{1}{t^2}\int_{1/\delta}^\infty|\varphi''(u)|du. \tag{49}$$

Using relation (15) and taking into account that

$$\lim_{u\to\infty}\psi(u)=0 \qquad \text{and} \qquad \lim_{u\to\infty}u\psi'(u)=0,$$

we get

$$\frac{1}{t^2}\int\limits_{1/\delta}^{\infty}|\varphi''(u)|du \le -\frac{2}{t^2\psi(\delta)}\int\limits_{1/\delta}^{\infty}d\psi(\delta u)+\frac{\delta}{t^2\psi(\delta)}\int\limits_{1/\delta}^{\infty}ud\psi'(\delta u)=\frac{3\psi(1)-\psi'(1)}{t^2\psi(\delta)}.$$

Using this relation and (49), we obtain

$$\left|\int\limits_0^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|\le\frac{K_1}{t^2\psi(\delta)},$$

whence

$$\int\limits_{|t|\ge\delta}\left|\int\limits_0^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt\le\frac{2K_1}{\delta\psi(\delta)}. \tag{50}$$

Let us estimate the integral $A(\varphi)$ on the segment $[-\delta,\delta]$. Since condition (44) is satisfied, we have

$$\int\limits_{-\delta}^{\delta}\left|\int\limits_0^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt$$

$$\le 2\delta\int\limits_0^{\infty}|\varphi(u)|du=\frac{\psi(1)}{\delta\psi(\delta)}+\frac{2\delta}{\psi(\delta)}\int\limits_{1/\delta}^{\infty}u\psi(\delta u)du$$

$$=\frac{\psi(1)}{\delta\psi(\delta)}+\frac{2}{\delta\psi(\delta)}\int\limits_1^{\infty}u\psi(u)du\le\frac{K_2}{\delta\psi(\delta)}. \tag{51}$$

Using relations (50) and (51), we conclude that the following estimate holds as $\delta\to\infty$:

$$A(\varphi)=O\left(\frac{1}{\delta\psi(\delta)}\right).$$

Thus, the transform $\hat{\varphi}(t)$ (11) is summable on the entire number axis.

We now establish the convergence of the integral $A(\nu)$ [see (14)], where $\hat{\nu}(t)$ is the transform (12) of the function $\nu(\cdot)$ defined by relation (10). To this end, we divide the set $(-\infty,\infty)$ into the two parts $[-\delta,\delta]$ and $|t|>\delta$ so that

$$A(\nu) = \frac{1}{\pi}\int\limits_{-\delta}^{\delta}\left|\int\limits_0^{\infty}\nu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt + \frac{1}{\pi}\int\limits_{|t|>\delta}\left|\int\limits_0^{\infty}\nu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt := I_1 + I_2. \quad (52)$$

Let us estimate the integral

$$I_1 = \frac{1}{\pi}\int\limits_{-\delta}^{\delta}\left|\int\limits_0^{\infty}\nu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt.$$

We have

$$I_1 \le \frac{1}{\pi}\int\limits_{-\delta}^{\delta}\left|\int\limits_0^{1/\delta}\nu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt + \frac{1}{\pi}\int\limits_{-\delta}^{\delta}\left|\int\limits_{1/\delta}^{\infty}\nu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt. \quad (53)$$

Taking inequality (35) into account, we obtain

$$\frac{1}{\pi}\int\limits_{-\delta}^{\delta}\left|\int\limits_0^{1/\delta}\nu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt \le \frac{1}{\pi}\int\limits_{-\delta}^{\delta}\int\limits_0^{1/\delta}|\nu(u)|\,du dt$$

$$= \frac{2\delta\psi(1)}{\pi\psi(\delta)}\int\limits_0^{1/\delta}\left(e^{-u}+u-1\right)du \le \frac{\psi(1)}{3\pi\delta^2\psi(\delta)}. \quad (54)$$

According to the conditions of Lemma 2, we have

$$\int\limits_1^{\infty}u^2\psi(u)du < \infty.$$

Using inequality (35) once again, we obtain the following estimate for the second integral on the right-hand side of (53):

$$\frac{1}{\pi}\int\limits_{-\delta}^{\delta}\left|\int\limits_{1/\delta}^{\infty}\nu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt \le \frac{1}{\pi}\int\limits_{-\delta}^{\delta}\int\limits_{1/\delta}^{\infty}|\nu(u)|dudt = \frac{2\delta}{\pi\psi(\delta)}\int\limits_{1/\delta}^{\infty}\left(e^{-u}+u-1\right)\psi(\delta u)du$$

$$\le \frac{\delta}{\pi\psi(\delta)}\int\limits_{1/\delta}^{\infty}u^2\psi(\delta u)du \le \frac{K}{\pi\delta^2\psi(\delta)}. \quad (55)$$

It follows from relations (53)–(55) that

$$I_1 = O\left(\frac{1}{\delta^2\psi(\delta)}\right), \quad \delta\to\infty. \quad (56)$$

Let us estimate the integral

$$I_2 = \frac{1}{\pi} \int\limits_{|t|>\delta} \left| \int\limits_0^\infty \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt.$$

Integrating twice by parts and taking into account that $\nu(0) = 0$ and $\nu'(0) = 0$, we get

$$\int\limits_0^{1/\delta} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du$$

$$= \frac{1}{t}\nu\left(\frac{1}{\delta}\right) \sin\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) + \frac{1}{t^2}\nu'\left(\frac{1}{\delta} - 0\right) \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \int\limits_0^{1/\delta} \nu''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du. \quad (57)$$

Taking into account that

$$\lim_{u\to\infty} \nu(u) = 0 \qquad \text{and} \qquad \lim_{u\to\infty} \nu'(u) = 0,$$

which follows from (16) and (17), we obtain

$$\int\limits_{1/\delta}^\infty \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du$$

$$= -\frac{1}{t}\nu(\frac{1}{\delta}) \sin\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2}\nu'\left(\frac{1}{\delta}\right) \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \int\limits_{1/\delta}^\infty \nu''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du. \quad (58)$$

Combining (57) and (58), we get

$$\int\limits_0^\infty \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du$$

$$= \frac{1}{t^2}\left(\frac{1}{\delta} + e^{-1/\delta} - 1\right) \frac{\delta\psi'(1)}{\psi(\delta)} \cos\left(\frac{t}{\delta} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2}\left[\int\limits_0^{1/\delta} + \int\limits_{1/\delta}^\infty\right] \nu''(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du.$$

Using inequality (35), we obtain

$$\left| \int\limits_0^\infty \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| \le \frac{1}{t^2}\left(\frac{K}{\delta\psi(\delta)} + \int\limits_0^{1/\delta} |\nu''(u)| du + \int\limits_{1/\delta}^\infty |\nu''(u)| du\right). \quad (59)$$

Since

$$\nu''(u) = -e^{-u}\frac{\psi(1)}{\psi(\delta)} \qquad \text{for} \quad u \in \left[0, \frac{1}{\delta}\right),$$

we get

$$\frac{1}{t^2}\int\limits_0^{1/\delta} |\nu''(u)|du = \frac{\psi(1)}{t^2\psi(\delta)}\int\limits_0^{1/\delta} e^{-u}du \le \frac{\psi(1)}{t^2\delta\psi(\delta)}. \tag{60}$$

For the estimation of the second integral on the right-hand side of (59), we use relations (27), (16), and (17). As a result, we obtain

$$\frac{1}{t^2}\int\limits_{1/\delta}^{\infty} |\nu''(u)|du \le \frac{1}{t^2\psi(\delta)}\left(\int\limits_{1/\delta}^{\infty}\psi(\delta u)du - 2\int\limits_{1/\delta}^{\infty} u d\psi(\delta u) + \frac{\delta}{2}\int\limits_{1/\delta}^{\infty} u^2 d\psi'(\delta u)\right)$$

$$= \frac{1}{t^2\psi(\delta)}\left(\int\limits_{1/\delta}^{\infty}\psi(\delta u)du - 2\left(\lim_{u\to\infty} u\psi(\delta u) - \frac{\psi(1)}{\delta} - \int\limits_{1/\delta}^{\infty}\psi(\delta u)du\right)\right.$$

$$\left. + \frac{\delta}{2}\left(\lim_{u\to\infty} u^2\psi'(\delta u) - \frac{\psi'(1)}{\delta^2}\right) - \int\limits_{1/\delta}^{\infty} u d\psi(\delta u)\right)$$

$$= \frac{1}{t^2\psi(\delta)}\left(4\int\limits_{1/\delta}^{\infty}\psi(\delta u)du + \frac{3\psi(1)}{\delta} - \frac{\psi'(1)}{2\delta}\right).$$

Since

$$\int\limits_1^{\infty}\psi(u)du < \infty,$$

we have

$$\frac{1}{t^2}\int\limits_{1/\delta}^{\infty} |\nu''(u)|du \le \frac{K}{t^2\delta\psi(\delta)}. \tag{61}$$

It follows from relations (59)–(61) that

$$\left|\int\limits_0^{\infty}\nu(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right| \le \frac{K_1}{t^2\delta\psi(\delta)}.$$

Then the following relation holds as $\delta \to \infty$:

$$I_2 = \frac{1}{\pi} \int\limits_{|t|>\delta} \left| \int\limits_0^{\infty} \nu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt = O\left(\frac{1}{\delta^2 \psi(\delta)}\right). \tag{62}$$

Combining relations (52), (56), and (62), we get

$$A(\nu) = O\left(\frac{1}{\delta^2 \psi(\delta)}\right), \quad \delta \to \infty. \tag{63}$$

Lemma 2 is proved.

Proof of Theorem 2. Lemma 2 states that integrals (13) and (14) are summable under the conditions of Theorem 2. Therefore, using relation (43) and taking estimate (63) into account, we obtain equality (45).

Theorem 2 is proved.

Examples of functions satisfying the conditions of Theorem 2 are functions $\psi \in \mathfrak{M}$ that have the following form for $t \geq 1$:

$$\psi(t) = \frac{\ln^{\alpha}(t+K)}{t^r}, \qquad \psi(t) = \frac{1}{t^r}(K + e^{-t}), \qquad r > 3, \quad K > 0, \quad \alpha \in R,$$

$$\psi(t) = t^r e^{-Kt^{\alpha}}, \qquad \psi(t) = \ln^r(t+e) e^{-Kt^{\alpha}}, \qquad K > 0, \quad \alpha > 0, \quad r \in R.$$

Assume that a function $\mu(\cdot)$ is associated with a function $\psi \in \mathfrak{M}$ by relation (2). Theorem 2 yields the following corollary:

Corollary 1. *If ψ belongs to $\mathfrak{M}_{\infty}$, the function $g(u)$ is convex downward for $u \in [b, \infty)$, $b \geq 1$, and*

$$\lim_{t\to\infty} \mu(\psi; t) = \infty, \tag{64}$$

then the asymptotic equality (45) holds as $\delta \to \infty$.

Proof. It suffices to verify that condition (64) guarantees the convergence of the integral

$$\int\limits_1^{\infty} u^2 \psi(u) du.$$

It follows from relations (12.24) in [6, p. 164] that the following inequality holds for any function $\psi \in \mathfrak{M}$:

$$\frac{\psi(t)}{|\psi'(t)|} \leq 2\,(\eta(t) - t) \quad \forall t \geq 1. \tag{65}$$

In view of (65), for any $r \geq 0$ one has

$$(t^r \psi(t))' = r t^{r-1} \psi(t) - t^r |\psi'(t)| \leq t^r |\psi'(t)| \left(2r \frac{\eta(t) - t}{t} - 1\right). \tag{66}$$

According to (64), the value $(\eta(t) - t)/t$ tends to zero as $t \to \infty$. Using relations (66), we conclude that, for any $r \geq 0$, there exists a number $t_0 = t_0(r, \psi)$ such that the function $t^r \psi(t)$ does not increase for $t > t_0$. Then

$$\int_1^\infty u^2 \psi(u) du = \int_1^\infty \frac{u^4 \psi(u)}{u^2} du \leq K \int_1^\infty \frac{du}{u^2} < \infty.$$

Thus, all conditions of Theorem 2 are satisfied. Therefore, equality (45) is true.

This work was supported by the Ukrainian State Foundation for Fundamental Research (grant No. 25.1/043).

REFERENCES

1. N. K. Bari, *Trigonometric Series* [in Russian], Fizmatgiz, Moscow (1961).
2. A. Zygmund, *Trigonometric Series* [Russian translation], Vol. 1, Mir, Moscow (1965).
3. A. I. Stepanets, *Classes of Periodic Functions and Approximation of Their Elements by Fourier Sums* [in Russian], Preprint No. 83.10, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1983).
4. A. I. Stepanets, "Deviations of Fourier sums on classes of infinitely differentiable functions," *Ukr. Mat. Zh.,* **36**, No. 6, 750–758 (1984).
5. A. I. Stepanets, *Classification and Approximation of Periodic Functions* [in Russian], Naukova Dumka, Kiev (1987).
6. A. I. Stepanets, *Methods of Approximation Theory* [in Russian], Vol. 1, Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev (2002).
7. T. V. Zhyhallo and Yu. I. Kharkevych, "Approximation of (ψ, β)-differentiable functions by Poisson integrals in the uniform metric," *Ukr. Mat. Zh.,* **61**, No. 11, 1497–1515 (2009).
8. L. I. Bausov, "Linear methods for summation of Fourier series with given rectangular matrices. I," *Izv. Vyssh. Uchebn. Zaved., Ser. Mat.,* **46**, No. 3, 15–31 (1965).