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APPROXIMATION OF CONJUGATE DIFFERENTIABLE FUNCTIONS BY BIHARMONIC POISSON INTEGRALS

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We determine the exact values of upper bounds of approximations by biharmonic Poisson integrals on classes of conjugate differentiable functions in uniform and integral metrics.

1. Main Definitions

Let C be the space of 2π-periodic continuous functions with the norm

$$\|f\|_C = \max_t |f(t)|,$$

let L_∞ be the space of 2π-periodic, measurable, essentially bounded functions with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_t |f(t)|,$$

and let L be the space of 2π-periodic functions summable on a period with the norm

$$\|f\|_L = \|f\|_1 = \int_{-\pi}^{\pi} |f(t)|dt.$$

We consider a boundary-value problem (in the unit disk) for the equation

$$\Delta(\Delta u) = 0, \tag{1}$$

where

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial x^2}$$

is the Laplace operator in polar coordinates.

By $B(\rho; f; x) = u(\rho, x)$ we denote a solution of Eq. (1) that satisfies the boundary conditions

$$u\,(\rho, x)\big|_{\rho=1} = f(x), \qquad \frac{\partial u(\rho, x)}{\partial\rho}\bigg|_{\rho=1} = 0, \quad -\pi \le x \le \pi, \tag{2}$$

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where $f(x)$ is a summable 2π-periodic function. Then a solution of the boundary-value problem (1), (2) can be represented in the form

$$B(\rho; f; x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left\{\frac{1}{2} + \sum_{k=1}^{\infty}\left[1 + \frac{k}{2}\left(1-\rho^2\right)\right]\rho^k \cos kt\right\}dt, \quad 0 \le \rho < 0. \tag{3}$$

Function (3) is called the biharmonic Poisson integral of f (see, e.g., [1]). Setting $\rho = e^{-1/\delta}$, we represent the biharmonic integral in the form

$$B_\delta(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)K_\delta(t)dt, \quad \delta > 0,$$

where

$$K_\delta(t) = \frac{1}{2} + \sum_{k=1}^{\infty}\left[1 + \frac{k}{2}\left(1 - e^{-2/\delta}\right)\right]e^{-k/\delta}\cos kt$$

is the biharmonic Poisson kernel.

Let W_p^r, where $p = 1$ or $p = \infty$, denote the set of 2π-periodic functions that have absolutely continuous derivatives up to the $(r-1)$th order inclusive and let $\|f^{(r)}(t)\|_p \le 1$ if $p = 1, \infty$. By $\overline{W}_p^r$ we denote the class of functions conjugate to functions from the class W_p^r, i.e.,

$$\overline{W}_p^r = \left\{\bar{f} : \bar{f}\,(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\,(x+t)\cot\frac{t}{2}dt, \;\; f \in W_p^r\right\},$$

where the integral is understood in the sense of its principal value, i.e.,

$$\int_{-\pi}^{\pi} f\,(x+t)\cot\frac{t}{2}dt = \lim_{\varepsilon\to 0+}\left(\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi}\right)f\,(x+t)\cot\frac{t}{2}dt$$

(see, e.g., [2, p. 22]).

Denote

$$\mathcal{E}\left(\mathfrak{N}, B_\delta\right)_C = \sup_{f\in\mathfrak{N}}\|f\,(x) - B_\delta\,(f, x)\|_C, \tag{4}$$

$$\mathcal{E}\left(\mathfrak{N}, B_\delta\right)_1 = \sup_{f\in\mathfrak{N}}\|f\,(x) - B_\delta\,(f, x)\|_1. \tag{5}$$

If a function $g\,(\delta) = g\,(\mathfrak{N}; \delta)$ such that the exact asymptotic equality

$$\mathcal{E}\left(\mathfrak{N}, B_\delta\right)_X = g\,(\delta) + o\,(g\,(\delta))$$

holds as $\delta \to \infty$ is determined in explicit form, then, following Stepanets [3, p. 198], we say that the Kolmogorov–Nikol'skii problem is solved for the class $\mathfrak{N}$ and operator $B_\delta(f, x)$ in the metric of the space X.

Approximating properties of the method of approximation by biharmonic Poisson integrals on classes of differentiable functions were studied by numerous mathematicians. In 1963, for $\mathcal{E}\left(W^1_\infty, B(\rho)\right)_C$, i.e., for the least upper bounds of deviations of functions from the class W^1_∞ from their biharmonic Poisson integrals, Kaniev [4] established the asymptotic equality $\big($for $\rho \to 1-\big)$

$$\mathcal{E}\left(W^1_\infty, B(\rho)\right)_C = \frac{2}{\pi}(1-\rho) + \frac{\varepsilon_\rho}{\pi}, \quad \varepsilon_\rho = o(1-\rho),$$

and determined the exact values of the approximating characteristics $\mathcal{E}\left(W^r_\infty, B(\rho)\right)_C$.

In 1968, Pych [5] obtained the asymptotic equality

$$\mathcal{E}\left(W^1_\infty, B(\rho)\right)_C = \frac{2}{\pi}(1-\rho) + O\left((1-\rho)^2 \ln\frac{1}{1-\rho}\right), \quad \rho \to 1-.$$

Later, these investigations were continued by Falaleev in [6], where the following complete asymptotic expansion of $\mathcal{E}\left(W^1_\infty, B(\rho)\right)_C$ was obtained:

$$\mathcal{E}\left(W^1_\infty, B(\rho)\right)_C = \frac{2}{\pi}\Bigg\{(1-\rho) + (1-\rho)^2 \ln\frac{1}{1-\rho} + \left(\ln 2 + \frac{1}{2}\right)(1-\rho)^2 + \sum_{k=3}^{\infty}\left(\alpha_k (1-\rho)^k \ln\frac{1}{1-\rho} + \beta_k (1-\rho)^k\right)\Bigg\},$$

$$\alpha_k = \frac{1}{k},$$

$$\beta_k = \frac{1}{k}\left(\ln 2 + \frac{1}{k} - \sum_{i=1}^{k-1}\frac{1}{i2^i} - \frac{1}{(k-2)(k-1)2^{k-2}} - \frac{1}{(k-1)2^{k-1}}\right).$$

In [7], Falaleev and Amanov obtained a complete asymptotic expansion of $\mathcal{E}\left(W^1_\infty, B_\delta\right)_C$ in terms of both $\frac{1}{\delta}$ and $1-\rho$, namely, the following relation holds as $\delta \to \infty$ $(\rho \to 1-0)$:

$$\mathcal{E}\left(W^1_\infty, B_\delta\right)_C = \frac{1-\rho^2}{\pi}\left\{1 + \sum_{k=1}^{\infty}(-1)^{k+1}\left(2k\int_\pi^\infty \frac{(t)_{2\pi}dt}{t^{2k+2}} - \frac{1}{\pi^{2k}}\right)\frac{1}{\delta^{2k}}\right\} + \left(\frac{2}{\pi}\frac{1}{\delta} - \frac{1-\rho^2}{\pi}\right)$$
$$\times\left\{\ln\delta + \ln\pi + \int_\pi^\infty \frac{(t)_{2\pi}dt}{t^2}\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{2k\pi^{2k}} - \int_\pi^\infty\frac{(t)_{2\pi}dt}{t^{2k+2}}\right)\frac{1}{\delta^{2k}}\right\},$$

where $(t)_{2\pi}$ is an even 2π-periodic extension of the function $\varphi(t) = t$ from $[0, \pi]$ to the entire axis. In the same work, general relations were obtained that enable one to deduce analogous expansions of $\mathcal{E}\left(W^r_\infty; B_\delta\right)_C$ for any $r \in N$.

The aim of the present paper is to determine the exact values of (4) and (5) for $\mathfrak{N} = \overline{W}^r_\infty$ and $\mathfrak{N} = \overline{W}^r_1$, $r \in N \setminus \{1\}$, in the uniform metric and in the integral metric, respectively.

Let K_n and $\tilde{K}_n$ denote the known Favard–Akhiezer–Krein constants from the theory of the best approximations, namely,

$$K_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m(n+1)}}{(2m+1)^{n+1}}, \quad n = 0, 1, 2, \ldots,$$

$$\tilde{K}_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{mn}}{(2m+1)^{n+1}}, \quad n \in N.$$

Theorem 1. *If $r = 2l$, $l \in N$, then the following equalities hold for every $\delta > 0$:*

$$\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C = \mathcal{E}\left(\overline{W}^r_1, B_\delta\right)_1$$

$$= \sum_{i=1}^{\frac{r}{2}}\frac{1}{(2i-1)!}K_{r-2i+1}\frac{1}{\delta^{2i-1}} - \sum_{i=1}^{\frac{r-2}{2}}\frac{1}{(2i)!}\tilde{K}_{r-2i}\frac{1}{\delta^{2i}}$$

$$+ \frac{1-e^{-2/\delta}}{2}\left(\sum_{i=1}^{\frac{r-2}{2}}\frac{1}{(2i-1)!}\tilde{K}_{r-2i}\frac{1}{\delta^{2i-1}} - \sum_{i=0}^{\frac{r-2}{2}}\frac{1}{(2i)!}K_{r-2i-1}\frac{1}{\delta^{2i}}\right)$$

$$- \alpha_\delta^{(r)} + \frac{1-e^{-2/\delta}}{2}\alpha_\delta^{(r-1)},$$

where

$$\alpha_\delta^{(n)} = \frac{2}{\pi}\int_0^{1/\delta}\int_0^{t_n}\ldots\int_0^{t_2}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1 dt_2\ldots dt_n.$$

Proof. First, we prove the theorem in the case of the uniform metric.

Integrating the Fourier coefficients of the function f r times by parts, we obtain

$$f(x) - B_\delta(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(x+t)\sum_{k=1}^{\infty}\frac{1-\left[1+\frac{k}{2}\left(1-e^{-2/\delta}\right)\right]e^{-k/\delta}}{k^r}\cos\left(kt+\frac{(r+1)\pi}{2}\right)dt. \quad (6)$$

Using the last equality, we get

$$\mathcal{E}(\overline{W}^r_\infty; B_\delta)_C = \frac{1}{\pi}\sup_{f\in W^r_\infty}\left|\int_{-\pi}^{\pi} f^{(r)}(t)\overline{Q}_r(t;\delta)\,dt\right|,$$

where

$$\overline{Q}_r(t;\delta) = \sum_{k=1}^{\infty} \frac{1 - \left[1 + \frac{k}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-k/\delta}}{k^r} \cos\left(kt + \frac{(r+1)\pi}{2}\right), \quad \delta > 0. \tag{7}$$

Since $f \in W^r_\infty$ and $\overline{Q}_r(t;\delta)$ is odd for $r = 2l,\ l \in N,$ we conclude that

$$\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C \le \frac{2}{\pi} \int_0^\pi \left|\overline{Q}_r(t;\delta)\right| dt.$$

Let us verify that

$$\operatorname{sign} \overline{Q}_r(t;\delta) = \pm \operatorname{sign} \sin t, \quad r = 2l, \quad l \in N. \tag{8}$$

It is obvious that

$$\overline{Q}_r(0;\delta) = \overline{Q}_r(\pi;\delta) = 0, \quad r = 2l, \quad l \in N.$$

Assume that there exists $t_0 \in (0,\pi)$ such that $\overline{Q}_r(t_0;\delta) = 0$. Then, applying the Rolle theorem $r-2$ times, we establish that, for the function $\overline{Q}_2(t;\delta)$, there exists a point $t_{r-2} \in (0,\pi)$ such that $\overline{Q}_2(t_{r-2};\delta) = 0,$ which is impossible because it follows from the remark to Theorem 1.14 in [8, p. 297] that $\overline{Q}_2(t;\delta) > 0,\ t \in (0,\pi)$. Thus, equality (8) is proved.

Consider a function f such that $f^{(r)}(t) = \operatorname{sign}\left(\overline{Q}_r(t;\delta)\right),\ t \in [-\pi,\pi]$. This function can be continuously and periodically extended to R and belongs to the class W^r_∞ [9, pp. 104–106]. Thus, for $r = 2l,\ l \in N,$ we have

$$\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C \ge \frac{2}{\pi} \int_0^\pi \left|\overline{Q}_r(t;\delta)\right| dt.$$

Therefore,

$$\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C = \frac{2}{\pi} \int_0^\pi \left|\overline{Q}_r(t;\delta)\right| dt = \frac{2}{\pi} \left| \int_0^\pi \overline{Q}_r(t;\delta)\, dt \right|. \tag{9}$$

According to (9), we get

$$\mathcal{E}(\overline{W}^r_\infty; B_\delta)_C = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1 - \left[1 + \frac{2k+1}{2}(1 - e^{-2/\delta})\right] e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}. \tag{10}$$

We rewrite equality (10) in the form

$$\mathcal{E}\left(\overline{W}^r_\infty; B_\delta\right)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-e^{-(2k+1)/\delta}}{(2k+1)^{r+1}} - \frac{2}{\pi}(1-e^{-2/\delta})\sum_{k=0}^{\infty}\frac{1}{(2k+1)^r}$$

$$+\frac{2}{\pi}\left(1-e^{-2/\delta}\right)\sum_{k=0}^{\infty}\frac{1-e^{-(2k+1)/\delta}}{(2k+1)^r}. \qquad (11)$$

We introduce the following function defined on $[0,\infty)$:

$$\varphi_n(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-e^{-(2k+1)/x}}{(2k+1)^{n+1}}, \quad n \geq 1.$$

This function admits the representation

$$\varphi_n(x) = \frac{2}{\pi}\int\limits_0^{1/x}\int\limits_{t_n}^{\infty}\dots\int\limits_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\dots dt_n,$$

and, in particular,

$$\varphi_1(x) = \frac{2}{\pi}\int\limits_0^{1/x}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1.$$

Indeed, since

$$\ln\frac{1+e^{-t_1}}{1-e^{-t_1}} = 2\sum_{k=0}^{\infty}\frac{e^{-(2k+1)t_1}}{2k+1},$$

we have

$$\frac{2}{\pi}\int\limits_0^{1/x}\int\limits_{t_n}^{\infty}\dots\int\limits_{t_3}^{\infty}\int\limits_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1dt_2\dots dt_{n-1}dt_n$$

$$= \frac{4}{\pi}\int\limits_0^{1/x}\int\limits_{t_n}^{\infty}\dots\int\limits_{t_3}^{\infty}\int\limits_{t_2}^{\infty}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)t_1}}{2k+1}dt_1dt_2\dots dt_{n-1}dt_n$$

$$= \frac{4}{\pi}\int\limits_0^{1/x}\int\limits_{t_n}^{\infty}\dots\int\limits_{t_3}^{\infty}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)t_2}}{(2k+1)^2}dt_2\dots dt_{n-1}dt_n$$

$$= \dots = \frac{4}{\pi}\int\limits_0^{1/x}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)t_n}}{(2k+1)^n}dt_n = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-e^{-(2k+1)/x}}{(2k+1)^{n+1}} = \varphi_n(x).$$

Performing certain transformations of the function $\varphi_n(x), \ n > 1,$ namely

$$\varphi_n(x) = \frac{2}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_n$$

$$= \frac{2}{\pi}\int_0^{1/x}\int_0^{\infty}\int_{t_{n-1}}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_n - \frac{2}{\pi}\int_0^{1/x}\int_0^{t_n}\int_{t_{n-1}}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_n$$

$$= \varphi_{n-1}(0)\int_0^{1/x}dt_n - \frac{2}{\pi}\int_0^{1/x}\int_0^{t_n}\int_{t_{n-1}}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_{n-1}dt_n,$$

and using the recurrence relations

$$\varphi_n(x) = \varphi_{n-1}(0)\int_0^{1/x}dt - \int_0^{1/x}\varphi_{n-1}\left(\frac{1}{t}\right)dt,$$

we get

$$\varphi_n(x) = \varphi_{n-1}(0)\int_0^{1/x}dt_1 - \int_0^{1/x}\varphi_{n-1}\left(\frac{1}{t_1}\right)dt_1$$

$$= \varphi_{n-1}(0)\int_0^{1/x}dt_1 - \varphi_{n-2}(0)\int_0^{1/x}\int_0^{t_1}dt_1dt_2 + \int_0^{1/x}\int_0^{t_1}\varphi_{n-2}\left(\frac{1}{t_2}\right)dt_1dt_2$$

$$= \ldots = \sum_{k=1}^{n-1}(-1)^{k-1}\varphi_{n-k}(0)\int_0^{1/x}\int_0^{t_1}\ldots\int_0^{t_{k-1}}dt_1\ldots dt_k$$

$$+ (-1)^{n-1}\frac{2}{\pi}\int_0^{1/x}\int_0^{t_1}\ldots\int_0^{t_{n-2}}\varphi_1\left(\frac{1}{t_{n-1}}\right)dt_1\ldots dt_{n-1}$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} \varphi_{n-k}(0) \int_0^{1/x} \int_0^{t_1} \dots \int_0^{t_{k-1}} dt_1 \dots dt_k$$

$$+ (-1)^{n-1} \frac{2}{\pi} \int_0^{1/x} \int_0^{t_n} \dots \int_0^{t_2} \ln \frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1 \dots dt_{n-1} dt_n,$$

i.e.,

$$\varphi_n(x) = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} \varphi_{n-k}(0) \frac{1}{x^k} + (-1)^{n-1} \alpha_x^{(n)}, \tag{12}$$

where

$$\varphi_n(0) = \begin{cases} K_n, & n = 2l-1, \\ \tilde{K}_n, & n = 2l, \end{cases} \qquad l \in N.$$

Taking into account the definition of the function $\varphi_n(x)$ and using equality (11), we obtain

$$\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C = \varphi_r(\delta) - \frac{1-e^{-2/\delta}}{2} \varphi_{r-1}(0) + \frac{1-e^{-2/\delta}}{2} \varphi_{r-1}(\delta).$$

Using relation (12), we get

$$\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C = \sum_{k=1}^{r-1} \frac{(-1)^{k-1}}{k!} \varphi_{r-k}(0) \frac{1}{\delta^k} - \alpha_\delta^{(r)} + \frac{1-e^{-2/\delta}}{2} \varphi_{r-1}(0)$$

$$+ \frac{1-e^{-2/\delta}}{2} \left(\sum_{k=1}^{r-2} \frac{(-1)^{k-1}}{k!} \varphi_{r-k-1}(0) \frac{1}{\delta^k} + \alpha_\delta^{(r-1)} \right)$$

$$= \sum_{i=1}^{\frac{r}{2}} \frac{1}{(2i-1)!} K_{r-2i+1} \frac{1}{\delta^{2i-1}} - \sum_{i=1}^{\frac{r-2}{2}} \frac{1}{(2i)!} \tilde{K}_{r-2i} \frac{1}{\delta^{2i}}$$

$$+ \frac{1-e^{-2/\delta}}{2} \left(\sum_{i=1}^{\frac{r-2}{2}} \frac{1}{(2i-1)!} \tilde{K}_{r-2i} \frac{1}{\delta^{2i-1}} - \sum_{i=0}^{\frac{r-2}{2}} \frac{1}{(2i)!} K_{r-2i-1} \frac{1}{\delta^{2i}} \right)$$

$$- \alpha_\delta^{(r)} + \frac{1-e^{-2/\delta}}{2} \alpha_\delta^{(r-1)}.$$

We have proved the theorem in the case of the uniform metric.

Let us show that $\mathcal{E}(\overline{W}^r_1; B_\delta)_1$ coincides with the right-hand side of (10), i.e., $\mathcal{E}(\overline{W}^r_\infty; B_\delta)_C = \mathcal{E}(\overline{W}^r_1; B_\delta)_1$.

Using equality (6), we obtain

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 = \sup_{f\in W_1^r} \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f^{(r)}(t+x)\,\overline{Q}_r(t;\delta)\,dt \right| dx, \quad r \in N, \tag{13}$$

where $\overline{Q}_r(t;\delta)$ is defined by (7).

By virtue of (8), the following relation holds for $r = 2l,\ l \in N$:

$$\mathcal{E}\left(\overline{W}_1^r; B_\delta\right)_1 \le \frac{1}{\pi} \int_{-\pi}^{\pi} \left|\overline{Q}_r(t;\delta)\right| dt = \frac{2}{\pi} \left| \int_0^{\pi} \sum_{k=1}^{\infty} \frac{1 - \left[1 + \frac{k}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-k/\delta}}{k^r} \sin kt\, dt \right|$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1 - \left[1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}. \tag{14}$$

On the other hand, using the lemma from [10, p. 63], for even r we get

$$\mathcal{E}\left(\overline{W}_1^r; B_\delta\right)_1 \ge \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1 - \left[1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}. \tag{15}$$

Comparing relations (14) and (15) and taking (10) into account, we conclude that the following relation holds for even r:

$$\mathcal{E}\left(\overline{W}_1^r; B_\delta\right)_1 = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1 - \left[1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-(2k+1)/\delta}}{(2k+1)^{r+1}} = \mathcal{E}\left(\overline{W}_\infty^r; B_\delta\right)_C.$$

Using the last equality, we conclude that the theorem is also true in the case of the integral metric.

Theorem 1 is proved.

Theorem 2. *If $r = 2l+1,\ l \in N$, then the following equalities hold for every $\delta > 0$:*

$$\mathcal{E}\left(\overline{W}_\infty^r, B_\delta\right)_C = \mathcal{E}\left(\overline{W}_1^r, B_\delta\right)_1$$

$$= \sum_{i=1}^{(r-1)/2} \frac{1}{(2i-1)!} K_{r-2i+1} \frac{1}{\delta^{2i-1}} - \sum_{i=1}^{(r-1)/2} \frac{1}{(2i)!} \tilde{K}_{r-2i} \frac{1}{\delta^{2i}}$$

$$+ \frac{1 - e^{-2/\delta}}{2} \left(\sum_{i=1}^{(r-1)/2} \frac{1}{(2i-1)!} \tilde{K}_{r-2i} \frac{1}{\delta^{2i-1}} - \sum_{i=0}^{(r-3)/2} \frac{1}{(2i)!} K_{r-2i-1} \frac{1}{\delta^{2i}} \right)$$

$$+ \beta_\delta^{(r)} - \frac{1 - e^{-2/\delta}}{2} \beta_\delta^{(r-1)},$$

where

$$\beta_\delta^{(r)} = \frac{4}{\pi}\int\limits_0^{1/\delta}\int\limits_0^{t_r}\ldots\int\limits_0^{t_2}\arctan e^{-t_1}dt_1\ldots dt_r.$$

Proof. First, we prove the theorem for the uniform metric.

Let $r=2l+1,\ l\in N.$ As in the proof of Theorem 1, we can show that

$$\mathcal{E}(\overline{W}^r_\infty;B_\delta)_C = \frac{1}{\pi}\sup_{f\in W^r_\infty}\left|\int\limits_{-\pi}^{\pi} f^{(r)}(t)\,\overline{Q}_r(t;\delta)\,dt\right|$$

$$= \frac{1}{\pi}\sup_{f\in W^r_\infty}\left|\int\limits_{-\pi}^{\pi} f^{(r)}(t)\left(\overline{Q}_r(t;\delta)-\overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right)dt\right|,$$

where $\overline{Q}_r(t;\delta)$ is defined by (7).

Since $f\in W^r_\infty$ and $\overline{Q}_r(t;\delta)$ is even for $r=2l+1,\ l\in N,$ we have

$$\mathcal{E}(\overline{W}^r_\infty;B_\delta)_C \le \frac{2}{\pi}\int\limits_0^{\pi}\left|\overline{Q}_r(t;\delta)-\overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right|dt.$$

Let us prove that

$$\operatorname{sign}\left(\overline{Q}_r(t;\delta)-\overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right)=\pm\operatorname{sign}\cos t,\quad r=2l+1,\quad l\in N. \tag{16}$$

Assume that

$$\overline{Q}_r(t_0;\delta)-\overline{Q}_r\left(\frac{\pi}{2};\delta\right)=0,\quad t_0\in(0,\pi),\quad t_0\neq\frac{\pi}{2}.$$

Then, according to the Rolle theorem, there exists a point $t_1\in(0,\pi)$ such that $\overline{Q}'_r(t_1;\delta)=0,$ whence $\overline{Q}_{r-1}(t_1;\delta)=0,$ which, by virtue of (8), is impossible. Equality (16) is proved.

Consider a function f such that

$$\operatorname{sign}\left(\overline{Q}_r(t;\delta)-\overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right)=\operatorname{sign}\cos t,\quad t\in[-\pi,\pi].$$

This function can be continuously and periodically extended to R and belongs to the class W^r_∞ [9, pp. 187, 188]. Thus, for $r=2l+1,\ l\in N,$ we get

$$\mathcal{E}(\overline{W}^r_\infty;B_\delta)_C \ge \frac{2}{\pi}\int\limits_0^{\pi}\left|\overline{Q}_r(t;\delta)-\overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right|dt,$$

and, therefore,

$$\mathcal{E}(\overline{W}^r_\infty; B_\delta)_C = \frac{2}{\pi}\int\limits_0^\pi \left|\overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right| dt$$

$$= \frac{2}{\pi}\left|\int\limits_0^{\pi/2}\left(\overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right)dt - \int\limits_0^{\pi/2}\left(\overline{Q}_r(\pi - t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right)dt\right|$$

$$= \frac{2}{\pi}\left|\int\limits_0^{\pi/2}\left(\overline{Q}_r(t;\delta) - \overline{Q}_r(\pi - t;\delta)\right)dt\right|. \tag{17}$$

Using (17), we obtain the following relation for $r = 2l+1,\ l \in N$:

$$\mathcal{E}\left(\overline{W}^r_\infty; B_\delta\right)_C = \frac{4}{\pi}\left|\int\limits_0^{\pi/2}\sum_{k=0}^\infty \frac{1 - \left[1 + \dfrac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right]e^{-(2k+1)/\delta}}{(2k+1)^r}\cos(2k+1)t\,dt\right|.$$

Thus, for $r = 2l+1,\ l \in N,$ we have

$$\mathcal{E}\left(\overline{W}^r_\infty; B_\delta\right)_C = \frac{4}{\pi}\sum_{k=0}^\infty (-1)^k \frac{1 - \left[1 + \dfrac{2k+1}{2}(1 - e^{-2/\delta})\right]e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}. \tag{18}$$

We rewrite (18) in the form

$$\mathcal{E}\left(\overline{W}^r_\infty; B_\delta\right)_C = \frac{4}{\pi}\sum_{k=0}^\infty (-1)^k \frac{1 - e^{-(2k+1)/\delta}}{(2k+1)^{r+1}} - \frac{2}{\pi}\left(1 - e^{-2/\delta}\right)\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^r}$$

$$+ \frac{2}{\pi}\left(1 - e^{-2/\delta}\right)\sum_{k=0}^\infty (-1)^k \frac{1 - e^{-(2k+1)/\delta}}{(2k+1)^r}. \tag{19}$$

We introduce the following function defined on $[0, \infty)$:

$$\psi_n(x) = \frac{4}{\pi}\sum_{k=0}^\infty (-1)^k \frac{1 - e^{-(2k+1)/x}}{(2k+1)^{n+1}}, \quad n \geq 1.$$

The function $\psi_n(x)$ admits the representation

$$\psi_n(x) = \frac{4}{\pi}\int\limits_0^{1/x}\int\limits_{t_n}^\infty \ldots \int\limits_{t_2}^\infty \arctan e^{-t_1}\,dt_1 \ldots dt_n,$$

and, in particular,

$$\psi_1(x) = \frac{4}{\pi} \int\limits_0^{1/x} \arctan e^{-t_1} dt_1.$$

Indeed, since

$$\arctan e^{-t_1} = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-(2k+1)t_1}}{2k+1},$$

we have

$$\frac{4}{\pi} \int\limits_0^{1/x} \int\limits_{t_n}^{\infty} \ldots \int\limits_{t_3}^{\infty} \int\limits_{t_2}^{\infty} \arctan e^{-t_1} dt_1 dt_2 \ldots dt_{n-1} dt_n$$

$$= \frac{4}{\pi} \int\limits_0^{1/x} \int\limits_{t_n}^{\infty} \ldots \int\limits_{t_3}^{\infty} \int\limits_{t_2}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{e^{-(2k+1)t_1}}{2k+1} dt_1 dt_2 \ldots dt_{n-1} dt_n$$

$$= \frac{4}{\pi} \int\limits_0^{1/x} \int\limits_{t_n}^{\infty} \ldots \int\limits_{t_3}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{e^{-(2k+1)t_2}}{(2k+1)^2} dt_2 \ldots dt_{n-1} dt_n$$

$$= \ldots = \frac{4}{\pi} \int\limits_0^{1/x} \sum_{k=0}^{\infty} \frac{e^{-(2k+1)t_n}}{(2k+1)^n} dt_n = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1 - e^{-(2k+1)/x}}{(2k+1)^{n+1}} = \psi_n(x).$$

We transform the function $\psi_n(x)$, $n > 1$, as follows:

$$\psi_n(x) = \frac{4}{\pi} \int\limits_0^{1/x} \int\limits_{t_n}^{\infty} \ldots \int\limits_{t_2}^{\infty} \arctan e^{-t_1} dt_1 \ldots dt_n$$

$$= \frac{4}{\pi} \int\limits_0^{1/x} \int\limits_0^{\infty} \int\limits_{t_{n-1}}^{\infty} \ldots \int\limits_{t_2}^{\infty} \arctan e^{-t_1} dt_1 \ldots dt_n - \frac{4}{\pi} \int\limits_0^{1/x} \int\limits_0^{t_n} \int\limits_{t_{n-1}}^{\infty} \ldots \int\limits_{t_2}^{\infty} \arctan e^{-t_1} dt_1 \ldots dt_n$$

$$= \psi_{n-1}(0) \int\limits_0^{1/x} dt - \frac{4}{\pi} \int\limits_0^{1/x} \int\limits_0^{t_n} \int\limits_{t_{n-1}}^{\infty} \ldots \int\limits_{t_2}^{\infty} \arctan e^{-t_1} dt_1 \ldots dt_n.$$

By using the recurrence relations

$$\psi_n(x) = \psi_{n-1}(0) \int\limits_0^{1/x} dt - \int\limits_0^{1/x} \psi_{n-1}\left(\frac{1}{t}\right) dt,$$

we obtain

$$\psi_n(x) = \psi_{n-1}(0) \int\limits_0^{1/x} dt_1 - \int\limits_0^{1/x} \psi_{n-1}\left(\frac{1}{t_1}\right) dt_1$$

$$= \psi_{n-1}(0) \int\limits_0^{1/x} dt_1 - \psi_{n-2}(0) \int\limits_0^{1/x}\int\limits_0^{t_1} dt_1 dt_2 + \int\limits_0^{1/x}\int\limits_0^{t_1} \psi_{n-2}\left(\frac{1}{t_2}\right) dt_1 dt_2$$

$$= \ldots = \sum_{k=1}^{n-1} (-1)^{k-1} \psi_{n-k}(0) \int\limits_0^{1/x}\int\limits_0^{t_1} \ldots \int\limits_0^{t_{k-1}} dt_1 \ldots dt_k$$

$$+ (-1)^{n-1} \frac{2}{\pi} \int\limits_0^{1/x}\int\limits_0^{t_1} \ldots \int\limits_0^{t_{n-2}} \psi_1\left(\frac{1}{t_{n-1}}\right) dt_1 \ldots dt_{n-1}$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} \psi_{n-k}(0) \int\limits_0^{1/x}\int\limits_0^{t_1} \ldots \int\limits_0^{t_{k-1}} dt_1 \ldots dt_k$$

$$+ (-1)^{n-1} \frac{4}{\pi} \int\limits_0^{1/x}\int\limits_0^{t_n} \ldots \int\limits_0^{t_2} \arctan e^{-t_1} dt_1 \ldots dt_{n-1} dt_n,$$

i.e.,

$$\psi_n(x) = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} \psi_{n-k}(0) \frac{1}{x^k} + (-1)^{n-1} \beta_x^{(n)}, \tag{20}$$

where

$$\psi_n(0) = \begin{cases} K_n, & n = 2l, \\ \tilde{K}_n, & n = 2l+1, \end{cases} \qquad l \in N.$$

Taking into account the definition of the function $\psi_n(x)$ and using equality (19), we get

$$\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C = \psi_r(\delta) - \frac{1 - e^{-2/\delta}}{2} \psi_{r-1}(0) + \frac{1 - e^{-2/\delta}}{2} \psi_{r-1}(\delta).$$

Using relation (20), we obtain

$$\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C = \sum_{k=1}^{r-1} \frac{(-1)^{k-1}}{k!}\psi_{r-k}(0)\frac{1}{\delta^k} + \beta_\delta^{(r)} - \frac{1-e^{-2/\delta}}{2}\psi_{r-1}(0)$$

$$+ \frac{1-e^{2/\delta}}{2}\left(\sum_{k=1}^{r-2} \frac{(-1)^{k-1}}{k!}\psi_{r-k-1}(0)\frac{1}{\delta^k} - \beta_\delta^{(r-1)}\right)$$

$$= \sum_{i=1}^{(r-1)/2} \frac{1}{(2i-1)!}K_{r-2i+1}\frac{1}{\delta^{2i-1}} - \sum_{i=1}^{(r-1)/2} \frac{1}{(2i)!}\tilde{K}_{r-2i}\frac{1}{\delta^{2i}}$$

$$+ \frac{1-e^{-2/\delta}}{2}\left(\sum_{i=1}^{(r-1)/2} \frac{1}{(2i-1)!}\tilde{K}_{r-2i}\frac{1}{\delta^{2i-1}} - \sum_{i=0}^{(r-3)/2} \frac{1}{(2i)!}K_{r-2i-1}\frac{1}{\delta^{2i}}\right)$$

$$+ \beta_\delta^{(r)} - \frac{1-e^{-2/\delta}}{2}\beta_\delta^{(r-1)},$$

i.e., the theorem is proved in the case of the uniform metric.

To prove this theorem in the case of the integral metric, it is necessary to prove the equality $\mathcal{E}\left(\overline{W}^r_\infty, B_\delta\right)_C = \mathcal{E}\left(\overline{W}^r_1, B_\delta\right)_1$.

Let us show that $\mathcal{E}\left(\overline{W}^r_1; B_\delta\right)_1$ coincides with the right-hand side of equality (18). Using equality (13) and the Fubini theorem [11, p. 331], for $r = 2l+1,\ l \in N,$ we get

$$\mathcal{E}\left(\overline{W}^r_1; B_\delta\right)_1 = \sup_{f \in W_1^r} \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} f^{(r)}(x+t)\left(\overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right)dt\right|dx$$

$$\le \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right|dt$$

$$= \frac{2}{\pi}\left|\left(\int_0^{\pi/2} - \int_{\pi/2}^{\pi}\right)\left(\overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right)dt\right|$$

$$= \frac{4}{\pi}\left|\int_0^{\pi/2}\sum_{k=0}^{\infty}\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}\cos(2k+1)t\,dt\right|$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}. \tag{21}$$

On the other hand, according to the lemma in [10, p. 63], the following relation holds for odd r:

$$\mathcal{E}\left(\overline{W}_1^r; B_\delta\right)_1 \geq \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{1-\left[1+\dfrac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}. \tag{22}$$

Using relations (21), (22), and (18) for $r = 2l+1,\ l \in N,$ we get

$$\mathcal{E}\left(\overline{W}_1^r; B_\delta\right)_1 = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{1-\left[1+\dfrac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-(2k+1)/\delta}}{(2k+1)^{r+1}} = \mathcal{E}\left(\overline{W}_\infty^r; B_\delta\right)_C.$$

Theorem 2 is proved.

It should be noted that values (4) and (5) for the classes $\mathfrak{N} = \overline{W}_\infty^r$ and $\mathfrak{N} = \overline{W}_1^r$, respectively, with the Abel–Poisson integral

$$P_\delta(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left(\frac{1}{2}+\sum_{k=1}^{\infty} e^{-k/\delta}\cos kt\right)dt, \quad \delta > 0,$$

instead of $B_\delta(f,x)$ were investigated in [12].

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