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APPROXIMATION OF CONJUGATE DIFFERENTIABLE FUNCTIONS BY THEIR ABEL–POISSON INTEGRALS

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We obtain the exact values of upper bounds of approximations of classes of periodic conjugate differentiable functions by their Abel–Poisson integrals in uniform and integral metrics.

Let C be the space of 2π-periodic continuous functions in which the norm is defined by the equality

$$\| f \|_C = \max_t | f(t) |,$$

let L_∞ be the space of 2π-periodic, continuous, measurable, essentially bounded functions with the norm

$$\| f \|_\infty = \operatorname{ess\,sup}_t | f(t) |,$$

and let L be the space of 2π-periodic functions summable on a period with the norm

$$\| f \|_L = \| f \|_1 = \int_0^{2\pi} | f(t) | dt.$$

Let W_p^r $(p = 1$ and $p = \infty)$ denote the set of 2π-periodic functions f having absolutely continuous derivatives up to the $(r-1)$th order inclusive and such that $\left\| f^{(r)} \right\|_p \leq 1$, $p = 1, \infty$, and let $\overline{W}_p^r$ denote the class of functions conjugate to functions from the class W_p^r, i.e.,

$$\overline{W}_p^r = \left\{ \bar{f} : \bar{f}(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \operatorname{ctg} \frac{t}{2} dt,\ f \in W_p^r \right\}. \tag{1}$$

Further, let $\Lambda = \{\lambda_\delta(k)\}$ denote the set of functions of natural argument dependent on a parameter δ (which varies on a certain set $E_\Lambda \subseteq R$ that has at least one limit point) and such that $\lambda_\delta(0) = 1 \ \ \forall \delta \in E_\Lambda$. Note that if $\delta \in N$, then the numbers $\lambda_\delta(k) =: \lambda_{n,k}$ are elements of an infinite rectangular matrix $\Lambda = \{\lambda_{n,k}\}$, $n, k = 0, 1, \ldots,$ $\lambda_{n,0} = 1$, $n \in N \cup \{0\}$. Using the set $\{\lambda_\delta(k)\}$, we associate every function $f(x)$ with the series

$$\frac{a_0}{2} \lambda_\delta(0) + \sum_{k=1}^{\infty} \lambda_\delta(k) \left(a_k \cos(kx) + b_k \sin(kx) \right), \quad \delta \in E_\Lambda,$$

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where a_0, a_k, and b_k are the Fourier coefficients of the function f. If, for $\lambda_\delta \in \Lambda$, $\delta \in E_\Lambda$, this series is the Fourier series of a certain continuous function, then we denote the latter by $U_\delta(f; x; \Lambda)$, and if $\delta \in N \cup \{0\}$, then we denote it by $U_n(f; x; \Lambda)$.

Under the condition that the sequence $\{\lambda_\delta(k)\}_{k=\overline{0,\infty}}$ is such that the series

$$K_\delta(t; \Lambda) = \frac{1}{2} + \sum_{k=1}^{\infty} \lambda_\delta(k) \cos kt \tag{2}$$

is the Fourier series of a certain summable function, by analogy with [1, p. 46] one can show that

$$U_\delta(f; x; \Lambda) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_\delta(t; \Lambda)\, dt. \tag{3}$$

Following Stepanets [2, p. 198], we call the problem of finding asymptotic equalities for the quantity

$$\mathcal{E}(\mathfrak{N}; U_\delta(f, \Lambda))_X = \sup_{f \in \mathfrak{N}} \| f(x) - U_\delta(f; x; \Lambda) \|_X \tag{4}$$

the *Kolmogorov–Nikol'skii problem*. Here, X is a normed space, $\mathfrak{N} \subseteq X$ is a given class of functions, and $U_\delta(f; x; \Lambda)$, $\delta \in E_\Lambda$, are operators generated by a certain method $U_\delta(f, \Lambda)$ of summation of Fourier series. If a function $\varphi(\delta) = \varphi(\mathfrak{N}; U_\delta(f, \Lambda); \delta)$ such that, as $\delta \to \delta_0$, where δ_0 is the limit point of the set E_Λ, one has

$$\mathcal{E}(\mathfrak{N}; U_\delta(f, \Lambda))_X = \varphi(\delta) + o(\varphi(\delta))$$

is determined in explicit form, then we say that the Kolmogorov–Nikol'skii problem is solved for the class $\mathfrak{N}$ and method $U_\delta(f, \Lambda)$.

In [3], Nikol'skii established the existence of a close relationship between the quantities $\mathcal{E}\big(W_1^r; U_n(\Lambda)\big)_1$ and $\mathcal{E}\big(W_\infty^r; U_n(\Lambda)\big)_C$ in the case where $\Lambda = \{\lambda_{n,k}\}$, $n = 0, 1, \ldots,$ $k = 0, 1, \ldots, n,$ is an arbitrary infinite triangular matrix. The investigations of Nikol'skii were continued by Stechkin and Telyakovskii in [4]. The most complete results for triangular Λ-methods of summation of Fourier series were obtained by Motornyi in [5].

In connection with operators generated by Λ-methods defined by the collection $\Lambda = \{\lambda_\delta(k)\}$ of functions continuous on $[0, \infty)$ and dependent on a real parameter δ, one should mention Pych's results [7], namely, the following lemmas:

Lemma 1. *If the function*

$$Q(t; \delta) = -\sum_{k=1}^{\infty} \frac{1 - \lambda_\delta(k)}{k} \sin kt$$

vanishes only at the points $t = k\pi$, $k = 0, \pm 1, \pm 2, \ldots,$ *for any* $\delta \in E_\Lambda$, *then the following equality holds for any integer* $r \geq 1$:

$$\mathcal{E}\big(W_\infty^r; U_\delta(\Lambda)\big)_C = \mathcal{E}\big(W_1^r; U_\delta(\Lambda)\big)_1.$$

Lemma 2. *If the function*

$$\bar{Q}(t;\delta) = \sum_{k=1}^{\infty} \frac{1-\lambda_\delta(k)}{k} \cos kt$$

has at most one root on the interval $(0, \pi]$, *then the following equality holds for integer* $r \geq 2$:

$$\mathscr{E}\left(\bar{W}_\infty^r; U_\delta(\Lambda)\right)_C = \mathscr{E}\left(\bar{W}_1^{\,r}; U_\delta(\Lambda)\right)_1.$$

If we set $\lambda_\delta(k) = e^{-k/\delta}$, $\delta > 0$, in (2), then operators of the type (3) are called Abel–Poisson integrals and are denoted by $P_\delta(f, x)$, i.e.,

$$P_\delta(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left(\frac{1}{2} + \sum_{k=1}^{\infty} e^{-k/\delta} \cos kt \right) dt. \tag{5}$$

The quantity $\bar{P}_\delta(f, x)$ is the conjugate Abel–Poisson integral, i.e,

$$\bar{P}_\delta(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{k=1}^{\infty} e^{-k/\delta} \sin kt \, dt. \tag{6}$$

In the case where $U_\delta(f; x; \Lambda) = P_\delta(f, x)$ and either $\mathfrak{R} = W_\infty^r$ or $\mathfrak{R} = \bar{W}_\infty^r$, quantities of the type (4) were studied in the uniform metric in [8–18].

The aim of the present paper is to determine the exact values of the following quantities for every $\delta > 0$:

$$\mathscr{E}\left(\bar{W}_\infty^r; P_\delta\right)_C = \sup_{f \in \bar{W}_\infty^r} \| f(x) - P_\delta(f, x) \|_C, \tag{7}$$

$$\mathscr{E}\left(\bar{W}_1^{\,r}; P_\delta\right)_1 = \sup_{f \in \bar{W}_1^{\,r}} \| f(x) - P_\delta(f, x) \|_1. \tag{8}$$

According to [10], quantity (7) satisfies the following equality for $r = 1$ and any $\delta > 0$:

$$\mathscr{E}\left(\bar{W}_\infty^1; P_\delta\right)_C = \frac{4}{\pi} \int_0^{1/\delta} \arctan e^{-t} \, dt.$$

As usual, we denote by K_n and $\tilde{K}_n$ the known Favard–Akhiezer–Krein constants from the theory of best approximations:

$$K_n = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m(n+1)}}{(2m+1)^{n+1}}, \qquad n = 0, 1, 2, \ldots,$$

$$\tilde{K}_n = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{mn}}{(2m+1)^{n+1}}, \qquad n \in N.$$

Theorem 1. *If* $r = 2l$, $l \in N$, *then the following equalities hold for every* $\delta > 0$:

$$\mathcal{E}\left(\overline{W}_\infty^r; P_\delta\right)_C = \mathcal{E}\left(\overline{W}_1^r; P_\delta\right)_1 = \sum_{i=1}^{r/2} \frac{1}{(2i-1)!} K_{r-2i+1} \frac{1}{\delta^{2i-1}} - \sum_{i=1}^{\frac{r-2}{2}} \frac{1}{(2i)!} \tilde{K}_{r-2i} \frac{1}{\delta^{2i}} - \alpha_\delta^{(r)}, \tag{9}$$

where

$$\alpha_\delta^{(r)} = \frac{2}{\pi} \int_0^{1/\delta} \int_0^{t_n} \ldots \int_0^{t_2} \ln \frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1\, dt_2 \ldots dt_n.$$

Proof. First, we prove the theorem in the case of the uniform metric. Taking (1) and (6) into account, we get

$$\bar{f}(x) - P_\delta(\bar{f}, x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) \left(\frac{1}{2} \operatorname{cotan} \frac{t}{2} - \sum_{k=1}^{\infty} e^{-k/\delta} \sin kt \right) dt.$$

Integrating r times by parts, we obtain

$$\bar{f}(x) - P_\delta(\bar{f}, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^{(r)}(t+x) \sum_{k=1}^{\infty} \frac{1-e^{-k/\delta}}{k^r} \cos\left(kt + \frac{(r+1)\pi}{2}\right) dt.$$

Therefore,

$$\mathcal{E}\left(\overline{W}_\infty^r; P_\delta\right)_C = \frac{1}{\pi} \sup_{f \in W_\infty^r} \left| \int_{-\pi}^{\pi} f^{(r)}(t)\, \overline{F}_{r,\delta}(t)\, dt \right|,$$

where

$$\overline{F}_{r,\delta}(t) = \sum_{k=1}^{\infty} \frac{1-e^{-k/\delta}}{k^r} \cos\left(kt + \frac{(r+1)\pi}{2}\right).$$

Since $f \in W_\infty^r$ and $\overline{F}_{r,\delta}(t)$ is odd for $r = 2l$, $l \in N$, we have

$$\mathcal{E}\left(\overline{W}_\infty^r; P_\delta\right)_C \le \frac{2}{\pi} \int_0^{\pi} \left| \overline{F}_{r,\delta}(t) \right| dt.$$

On the other hand, if $\operatorname{sign}\left(\overline{F}_{r,\delta}(t)\right) = \pm\operatorname{sign}\sin t$, then the function f such that $f^{(r)}(t) = \operatorname{sign}\left(\overline{F}_{r,\delta}(t)\right)$, $t \in [-\pi, \pi]$, is continuously and periodically extended to R and belongs to the class W_∞^r [6, pp. 104–106]. Thus, for $r = 2l$, $l \in N$, we have

$$\mathcal{E}\left(\overline{W}_\infty^r; P_\delta\right)_C \geq \frac{2}{\pi}\int_0^\pi \left|\overline{F}_{r,\delta}(t)\right| dt$$

whence

$$\mathcal{E}\left(\overline{W}_\infty^r; P_\delta\right)_C = \frac{2}{\pi}\int_0^\pi \left|\overline{F}_{r,\delta}(t)\right| dt = \frac{2}{\pi}\left|\int_0^\pi \overline{F}_{r,\delta}(t)\, dt\right|. \tag{10}$$

For $r = 2l$, $l \in N$, and $t \in [-\pi, \pi]$, the equality $\operatorname{sign}\left(\overline{F}_{r,\delta}(t)\right) = \pm\operatorname{sign}\sin t$ is established by using the following arguments:

It is obvious that $\overline{F}_{r,\delta}(0) = \overline{F}_{r,\delta}(\pi) = 0$ for $r = 2l$, $l \in N$. Assuming that $\overline{F}_{r,\delta}(t) = 0$ for a certain $t_0 \in (0, \pi)$ and using the Rolle theorem $r - 1$ times, one can conclude that, for the function

$$\overline{F}_{1,\delta}(t) = -\sum_{k=1}^{\infty} \frac{1 - e^{-k/\delta}}{k} \cos kt,$$

there exist $t_{r-1}^{(1)}, t_{r-1}^{(2)} \in (0, \pi)$, $t_{r-1}^{(1)} \neq t_{r-1}^{(2)}$, such that

$$\overline{F}_{1,\delta}\left(t_{r-1}^{(1)}\right) = \overline{F}_{1,\delta}\left(t_{r-1}^{(2)}\right) = 0.$$

However, this contradicts the fact that, according to relations (1.441.2) and (1.448.2) in [19], the function $\overline{F}_{1,\delta}(t)$ can be represented in the form

$$\overline{F}_{1,\delta}(t) = \frac{1}{2}\ln\frac{2(1-\cos t)}{1 - 2e^{-1/\delta}\cos t + e^{-2/\delta}}, \quad t \in (0, \pi).$$

It is easy to verify that the equation $\overline{F}_{1,\delta}(t) = 0$ has only one root on the interval $(0, \pi)$.

Thus, using (10), we obtain the following relation for $r = 2l$, $l \in N$, and $\delta > 0$:

$$\mathcal{E}\left(\overline{W}_\infty^r, P_\delta\right)_C = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1 - e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}.$$

We introduce the following function defined on $[0, \infty)$:

$$\varphi_n(x) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1 - e^{-(2k+1)/x}}{(2k+1)^{n+1}}, \qquad n \geq 1.$$

This function can be represented in the form

$$\varphi_n(x) = \frac{2}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_n;$$

in particular,

$$\varphi_1(x) = \frac{2}{\pi}\int_0^{1/x}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1.$$

Indeed, since

$$\ln\frac{1+e^{-t_1}}{1-e^{-t_1}} = 2\sum_{k=0}^{\infty}\frac{e^{-(2k+1)t_1}}{2k+1},$$

we have

$$\frac{2}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_3}^{\infty}\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\,dt_2\ldots dt_{n-1}\,dt_n = \frac{4}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_3}^{\infty}\int_{t_2}^{\infty}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)t_1}}{2k+1}dt_1\,dt_2\ldots dt_{n-1}\,dt_n$$

$$= \frac{4}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_3}^{\infty}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)t_2}}{(2k+1)^2}dt_2\ldots dt_{n-1}\,dt_n$$

$$= \ldots = \frac{4}{\pi}\int_0^{1/x}\sum_{k=0}^{\infty}\frac{e^{-(2k+1)t_n}}{(2k+1)^n}dt_n$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-e^{-(2k+1)/x}}{(2k+1)^{n+1}} = \varphi_n(x).$$

Performing transformations of the function $\varphi_n(x)$, $n>1$, namely

$$\varphi_n(x) = \frac{2}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_n$$

$$= \frac{2}{\pi}\int_0^{1/x}\int_0^{\infty}\int_{t_{n-1}}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_n - \frac{2}{\pi}\int_0^{1/x}\int_0^{t_n}\int_{t_{n-1}}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_n$$

$$= \varphi_{n-1}(0)\int_0^{1/x}dt_n - \frac{2}{\pi}\int_0^{1/x}\int_0^{t_n}\int_{t_{n-1}}^{\infty}\ldots\int_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_{n-1}\,dt_n,$$

and using the recurrence relations

$$\varphi_n(x) = \varphi_{n-1}(0)\int_0^{1/x} dt - \int_0^{1/x} \varphi_{n-1}\left(\frac{1}{t}\right)dt,$$

we get

$$\varphi_n(x) = \varphi_{n-1}(0)\int_0^{1/x} dt_1 - \int_0^{1/x} \varphi_{n-1}\left(\frac{1}{t_1}\right)dt_1$$

$$= \varphi_{n-1}(0)\int_0^{1/x} dt_1 - \varphi_{n-2}(0)\int_0^{1/x}\int_0^{t_1} dt_1\,dt_2 + \int_0^{1/x}\int_0^{t_1}\varphi_{n-2}\left(\frac{1}{t_2}\right)dt_1\,dt_2$$

$$= \ldots = \sum_{k=1}^{n-1}(-1)^{k-1}\varphi_{n-k}(0)\int_0^{1/x}\int_0^{t_1}\ldots\int_0^{t_{k-1}} dt_1\ldots dt_k + (-1)^{n-1}\frac{2}{\pi}\int_0^{1/x}\int_0^{t_1}\ldots\int_0^{t_{n-2}}\varphi_1\left(\frac{1}{t_{n-1}}\right)dt_1\ldots dt_{n-1}$$

$$= \sum_{k=1}^{n-1}(-1)^{k-1}\varphi_{n-k}(0)\int_0^{1/x}\int_0^{t_1}\ldots\int_0^{t_{k-1}} dt_1\ldots dt_k + (-1)^{n-1}\frac{2}{\pi}\int_0^{1/x}\int_0^{t_n}\ldots\int_0^{t_2}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1\ldots dt_{n-1}\,dt_n.$$

Therefore,

$$\varphi_n(x) = \sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k!}\varphi_{n-k}(0)\frac{1}{x^k} + (-1)^{n-1}\alpha_x^{(n)}, \tag{11}$$

where

$$\varphi_n(0) = \begin{cases} K_n, & n = 2l-1, \\ \tilde{K}_n, & n = 2l, \end{cases} \qquad l \in N.$$

For $r = 2l$, $l \in N$, we have

$$\mathcal{E}\left(\overline{W}{}_\infty^r, P_\delta\right)_C = \varphi_r(\delta) = \sum_{k=1}^{r-1}\frac{(-1)^{k-1}}{k!}\varphi_{r-k}(0)\frac{1}{\delta^k} - \alpha_\delta^{(r)}$$

$$= \sum_{i=1}^{r/2}\frac{1}{(2i-1)!}K_{r-2i+1}\frac{1}{\delta^{2i-1}} - \sum_{i=1}^{(r-2)/2}\frac{1}{(2i)!}\tilde{K}_{r-2i}\frac{1}{\delta^{2i}} - \alpha_\delta^{(r)}.$$

Thus, in the case of the uniform metric, equality (9) is true. For $p = 1$, relation (9) follows from Lemma 2 with regard for the fact that the function $\overline{Q}(t;\delta) = -\overline{F}_{1,\delta}(t)$ has only one root on the interval $(0, \pi]$.

Theorem 1 is proved.

Theorem 2. *If* $r = 2l + 1$, $l \in N$, *then the following equalities hold for every* $\delta > 0$:

$$\mathcal{E}\left(\overline{W}_\infty^r, P_\delta\right)_C = \mathcal{E}\left(\overline{W}_1^r, P_\delta\right)_1 = \sum_{i=1}^{(r-1)/2} \frac{1}{(2i-1)!} K_{r-2i+1} \frac{1}{\delta^{2i-1}} - \sum_{i=1}^{(r-1)/2} \frac{1}{(2i)!} \tilde{K}_{r-2i} \frac{1}{\delta^{2i}} + \beta_\delta^{(r)}, \qquad (12)$$

where

$$\beta_\delta^{(r)} = \frac{4}{\pi} \int_0^{1/\delta} \int_0^{t_r} \ldots \int_0^{t_2} \arctan e^{-t_1}\, dt_1 \ldots dt_r .$$

Proof. Let us show that relation (12) is true in the case of the uniform metric. Taking into account that

$$\int_{-\pi}^{\pi} f^{(r)}(t) dt = 0,$$

we get

$$\mathcal{E}\left(\overline{W}_\infty^r, P_\delta\right)_C = \frac{1}{\pi} \sup_{f \in W_\infty^r} \left| \int_{-\pi}^{\pi} f^{(r)}(t)\, \overline{F}_{r,\delta}(t)\, dt \right| = \frac{1}{\pi} \sup_{f \in W_\infty^r} \left| \int_{-\pi}^{\pi} f^{(r)}(t) \left(\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right)\right) dt \right| .$$

Since $f \in W_\infty^r$ and $\overline{F}_{r,\delta}(t)$ is even for $r = 2l + 1$, $l \in N$, we have

$$\mathcal{E}\left(\overline{W}_\infty^r, P_\delta\right)_C \le \frac{2}{\pi} \int_0^{\pi} \left| \overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right) \right| dt .$$

On the other hand, if

$$\operatorname{sign}\left(\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right)\right) = \pm \operatorname{sign} \cos t,$$

then the function f such that

$$f^{(r)}(t) = \operatorname{sign}\left(\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right)\right), \qquad t \in [-\pi, \pi],$$

is continuously and periodically extended to R and belongs to the class W_∞^r [6, pp. 187, 188]. Thus, for $r = 2l + 1$, $l \in N$, we have

$$\mathcal{E}\left(\overline{W}_\infty^r, P_\delta\right)_C \ge \frac{2}{\pi} \int_0^{\pi} \left| \overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right) \right| dt$$

and, hence,

$$\mathcal{E}\left(\overline{W}_{\infty}^{r}, P_{\delta}\right)_{C} = \frac{2}{\pi}\int_{0}^{\pi}\left|\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right)\right|dt$$

$$= \frac{2}{\pi}\left|\int_{0}^{\pi/2}\left(\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right)\right)dt - \int_{0}^{\pi/2}\left(\overline{F}_{r,\delta}(\pi - t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right)\right)dt\right|$$

$$= \frac{2}{\pi}\left|\int_{0}^{\pi/2}\left(\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}(\pi - t)\right)dt\right|. \tag{13}$$

The equality

$$\operatorname{sign}\left(\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right)\right) = \pm\operatorname{sign}\cos t$$

follows from the arguments presented below.

Under the assumption that

$$\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right) = 0, \qquad r = 2l + 1, \quad l \in N,$$

one can conclude that, for a certain $t_0 \in (0, \pi)$, $t_0 \neq \frac{\pi}{2}$, according to the Rolle theorem, there exists $t_1 \in (0, \pi)$ such that $\overline{F}'_{r,\delta}(t_1) = 0$, whence $\overline{F}_{r-1,\delta}(t_1) = 0$. However, this contradicts the fact that

$$\operatorname{sign}\left(\overline{F}_{r-1,\delta}(t)\right) = \pm\operatorname{sign}\sin t \qquad \text{for} \quad r = 2l + 1, \quad l \in N.$$

Thus, $t = \frac{\pi}{2}$ is the unique solution of the equation

$$\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right) = 0$$

on the segment $[0, \pi]$. Since

$$\operatorname{sign}\left(\overline{F}'_{r,\delta}(t)\right) = \pm\operatorname{sign}\sin t \qquad \text{for} \quad r = 2l + 1, \quad l \in N,$$

the function

$$\overline{F}_{r,\delta}(t) - \overline{F}_{r,\delta}\left(\frac{\pi}{2}\right)$$

is monotone on $(0, \pi)$.

Using relation (13), for $r = 2l + 1$, $l \in N$, we obtain

$$\mathcal{E}\left(\overline{W}_{\infty}^{r}, P_{\delta}\right)_C = \frac{4}{\pi}\left|\int_0^{\pi/2}\sum_{k=0}^{\infty}\frac{1-e^{-\frac{2k+1}{\delta}}}{(2k+1)^r}\cos(2k+1)t\,dt\right|.$$

Thus, for $r = 2l + 1$, $l \in N$, and $\delta > 0$, we get

$$\mathcal{E}\left(\overline{W}_{\infty}^{r}, P_{\delta}\right)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1-e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}.$$

We introduce the following function defined on $[0, \infty)$:

$$\psi_n(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1-e^{-(2k+1)/x}}{(2k+1)^{n+1}}, \qquad n \geq 1.$$

The function $\psi_n(x)$ admits the representation

$$\psi_n(x) = \frac{4}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_2}^{\infty}\arctan e^{-t_1}\,dt_1\ldots dt_n;$$

in particular,

$$\psi_1(x) = \frac{4}{\pi}\int_0^{1/x}\arctan e^{-t_1}\,dt_1.$$

Indeed, since

$$\arctan e^{-t_1} = \sum_{k=0}^{\infty}(-1)^k\frac{e^{-(2k+1)t_1}}{2k+1},$$

we have

$$\frac{4}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_3}^{\infty}\int_{t_2}^{\infty}\arctan e^{-t_1}\,dt_1 dt_2\ldots dt_{n-1}\,dt_n$$

$$= \frac{4}{\pi}\int_0^{1/x}\int_{t_n}^{\infty}\ldots\int_{t_3}^{\infty}\int_{t_2}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{e^{-(2k+1)t_1}}{2k+1}\,dt_1 dt_2\ldots dt_{n-1}\,dt_n$$

$$= \frac{4}{\pi} \int_0^{1/x} \int_{t_n}^{\infty} \ldots \int_{t_3}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{e^{-(2k+1)t_2}}{(2k+1)^2} \, dt_2 \ldots dt_{n-1} \, dt_n$$

$$= \ldots = \frac{4}{\pi} \int_0^{1/x} \sum_{k=0}^{\infty} \frac{e^{-(2k+1)t_n}}{(2k+1)^n} \, dt_n = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1 - e^{-(2k+1)/x}}{(2k+1)^{n+1}} = \psi_n(x).$$

We transform the function $\psi_n(x)$, $n > 1$, as follows:

$$\psi_n(x) = \frac{4}{\pi} \int_0^{1/x} \int_{t_n}^{\infty} \ldots \int_{t_2}^{\infty} \arctan e^{-t_1} \, dt_1 \ldots dt_n$$

$$= \frac{4}{\pi} \int_0^{1/x} \int_0^{\infty} \int_{t_{n-1}}^{\infty} \ldots \int_{t_2}^{\infty} \arctan e^{-t_1} \, dt_1 \ldots dt_n$$

$$- \frac{4}{\pi} \int_0^{1/x} \int_0^{t_n} \int_{t_{n-1}}^{\infty} \ldots \int_{t_2}^{\infty} \arctan e^{-t_1} \, dt_1 \ldots dt_n$$

$$= \psi_{n-1}(0) \int_0^{1/x} dt - \frac{4}{\pi} \int_0^{1/x} \int_0^{t_n} \int_{t_{n-1}}^{\infty} \ldots \int_{t_2}^{\infty} \arctan e^{-t_1} \, dt_1 \ldots dt_n .$$

Further, using the recurrence relations

$$\psi_n(x) = \psi_{n-1}(0) \int_0^{1/x} dt - \int_0^{1/x} \psi_{n-1}\left(\frac{1}{t}\right) dt,$$

we get

$$\psi_n(x) = \psi_{n-1}(0) \int_0^{1/x} dt_1 - \int_0^{1/x} \psi_{n-1}\left(\frac{1}{t_1}\right) dt_1$$

$$= \psi_{n-1}(0) \int_0^{1/x} dt_1 - \psi_{n-2}(0) \int_0^{1/x} \int_0^{t_1} dt_1 \, dt_2 + \int_0^{1/x} \int_0^{t_1} \psi_{n-2}\left(\frac{1}{t_2}\right) dt_1 \, dt_2$$

$$= \ldots = \sum_{k=1}^{n-1} (-1)^{k-1} \psi_{n-k}(0) \int_0^{1/x} \int_0^{t_1} \ldots \int_0^{t_{k-1}} dt_1 \ldots dt_k$$

$$+ (-1)^{n-1} \frac{2}{\pi} \int_0^{1/x} \int_0^{t_1} \ldots \int_0^{t_{n-2}} \psi_1\left(\frac{1}{t_{n-1}}\right) dt_1 \ldots dt_{n-1}$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} \psi_{n-k}(0) \int_0^{1/x} \int_0^{t_1} \ldots \int_0^{t_{k-1}} dt_1 \ldots dt_k$$

$$+ (-1)^{n-1} \frac{4}{\pi} \int_0^{1/x} \int_0^{t_n} \ldots \int_0^{t_2} \arctan e^{-t_1} \, dt_1 \ldots dt_{n-1} \, dt_n .$$

Hence,

$$\psi_n(x) = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} \psi_{n-k}(0) \frac{1}{x^k} + (-1)^{n-1} \beta_x^{(n)},$$

where

$$\psi_n(0) = \begin{cases} K_n, & n = 2l, \\ \tilde{K}_n, & n = 2l+1, \end{cases} \qquad l \in N .$$

Therefore, for $r = 2l + 1$, $l \in N$, we obtain

$$\mathcal{E}\left(\overline{W}_\infty^r, P_\delta\right)_C = \psi_r(\delta) = \sum_{k=1}^{r-1} \frac{(-1)^{k-1}}{k!} \psi_{r-k}(0) \frac{1}{\delta^k} + \beta_\delta^{(r)}$$

$$= \sum_{i=1}^{(r-1)/2} \frac{1}{(2i-1)!} K_{r-2i+1} \frac{1}{\delta^{2i-1}} - \sum_{i=1}^{(r-1)/2} \frac{1}{(2i)!} \tilde{K}_{r-2i} \frac{1}{\delta^{2i}} + \beta_\delta^{(r)}.$$

Thus, equality (12) is true in the case of the uniform metric. For $p = 1$, relation (12) follows from Lemma 2.

Theorem 2 is proved.

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