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APPROXIMATION OF FUNCTIONS FROM THE CLASS $\hat{C}^{\psi}_{\beta,\infty}$ BY POISSON BIHARMONIC OPERATORS IN THE UNIFORM METRIC

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We obtain asymptotic equalities for upper bounds of approximations of functions from the class $\hat{C}^{\psi}_{\beta,\infty}$ by Poisson biharmonic operators in the uniform metric.

Let $\hat{L}_1$ be the set of functions φ defined on the entire real axis R with the finite norm

$$\|\varphi\|_{\hat{1}} = \sup_{a\in R} \int\limits_{a}^{a+2\pi} |\varphi(t)|dt,$$

let $\hat{L}_\infty$ be the space of functions measurable and essentially bounded on the entire axis with the finite norm

$$\|\varphi\|_{\hat{\infty}} = \operatorname*{ess\,sup}_{t\in R} |\varphi(t)|,$$

and let $\hat{C}$ denote the set of functions continuous and defined on the real axis with the finite norm

$$\|f\|_{\hat{C}} = \sup_{x\in R} \big|f(x)\big|.$$

Stepanets' (see, e.g., [1, 2]) introduced classes $\hat{L}^{\psi}_{\beta}\mathfrak{N}$ of functions defined on the entire real axis as follows: Let $\beta \in R$ and a function $\psi(v)$ continuous for all $v \geq 0$ be such that the transform

$$\hat{\psi}(t) = \frac{1}{\pi} \int\limits_{0}^{\infty} \psi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv$$

is summable on the entire number axis. Let $\hat{L}^{\psi}_{\beta}$ denote the set of functions $f(x) \in \hat{L}_1$ that can be represented in the following form for almost all $x \in R$:

$$f(x) = A_0 + \int\limits_{-\infty}^{\infty} \varphi(x+t) \frac{1}{\pi} \int\limits_{0}^{\infty} \psi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dvdt, \tag{1}$$

where A_0 is a certain constant, $\varphi \in \hat{L}_1$, $\beta \in R$, and the integral is understood as the limit of integrals over increasing symmetric intervals.

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If $f \in \hat{L}^{\psi}_{\beta}$ and $\varphi \in \mathfrak{N}$, $\mathfrak{N} \subset \hat{L}_1$, then it is assumed that $f \in \hat{L}^{\psi}_{\beta}\mathfrak{N}$. Let $\hat{C}^{\psi}_{\beta}$ $(\hat{C}^{\psi}_{\beta}\mathfrak{N})$ denote the subset of continuous functions from $\hat{L}^{\psi}_{\beta}$ $(\hat{L}^{\psi}_{\beta}\mathfrak{N})$ and let

$$\hat{C}^{\psi}_{\beta,\infty} = \left\{ f \in \hat{C}^{\psi}_{\beta} \colon \|\varphi\|_{\hat{\infty}} \le 1 \right\}.$$

The function $\varphi(\cdot)$ in (1) is called the (ψ, β)-derivative of the function $f(\cdot)$ (see, e.g., [3, p. 170]) and is denoted by $f^{\psi}_{\beta}(\cdot)$.

Let $\mathfrak{M}$ denote (see [4, p. 93] or [5, p. 159]) the set of positive, continuous, convex downward functions $\psi(v)$, $v \ge 1$, for which

$$\lim_{v\to\infty} \psi(v) = 0.$$

Subsets $\mathfrak{M}_0$ and $\mathfrak{M}_C$ of the set $\mathfrak{M}$ are defined as follows (see, e.g., [5, p. 160]):

$$\mathfrak{M}_0 = \left\{ \psi \in \mathfrak{M} : 0 < \frac{t}{\eta(t) - t} \le K \ \forall t \ge 1 \right\}$$

and

$$\mathfrak{M}_C = \left\{ \psi \in \mathfrak{M} \colon 0 < K_1 \le \frac{t}{\eta(t) - t} \le K_2 \ \ \forall t \ge 1 \right\},$$

where

$$\eta(t) = \eta(\psi, t) = \psi^{-1}\left(\frac{1}{2}\psi(t)\right)$$

and ψ^{-1} is the function inverse to ψ. Here and in what follows, K and K_i denote constants, which, generally speaking, may be different in different relations.

We extend every function $\psi \in \mathfrak{M}$ to the interval $[0, 1)$ so that the following conditions are satisfied:

(i) the obtained function $\big($denoted, as before, by $\psi(v)\big)$ is continuous for all $v \ge 0$, $\psi(0) = 0$;

(ii) the derivative $\psi'(v) = \psi'(v + 0)$ has bounded variation on the interval $[0, \infty)$, and $\psi(v)$ has the continuous second derivative on $[0, \infty)$ everywhere except the point $v = 1$;

(iii) $\psi(v)$ is increasing and convex downward on $[0, 1]$.

Denote the set of these functions by $\mathfrak{A}$. Let $\mathfrak{A}'$ denote the subset of functions $\psi \in \mathfrak{A}$ for which

$$\int_1^{\infty} \frac{\psi(t)}{t}\, dt < \infty$$

and let

$$\mathfrak{A}_C = \left\{ \psi(v) \in \mathfrak{A} \colon \psi \in \mathfrak{M}_C, \ \ v \in [1, \infty) \right\}.$$

Let $\Lambda = \left\{\lambda_\sigma\left(\frac{v}{\sigma}\right)\right\}$ be a collection of functions continuous for all $v \geq 0$ and dependent on a real parameter σ. We associate every function $f \in \hat{L}^{\psi}_{\beta}$ with an expression of the form

$$U_\sigma(f;x;\Lambda) = A_0 + \int\limits_{-\infty}^{\infty} f^{\psi}_{\beta}(x+t)\,\frac{1}{\pi}\int\limits_{0}^{\infty}\psi(v)\lambda_\sigma\left(\frac{v}{\sigma}\right)\cos\left(vt+\frac{\beta\pi}{2}\right)dvdt,$$

where $\psi(v)$ is a function continuous for all $v \geq 0$ and $\beta \in R$. In the case where

$$\lambda_\sigma(v) = \left[1+\frac{v\sigma}{2}\left(1-e^{-\frac{2}{\sigma}}\right)\right]e^{-v}, \quad \sigma \in (0,\infty),$$

we denote the functions $U_\sigma(f;x;\Lambda)$ by $B_\sigma(f;x)$:

$$B_\sigma(f;x) = A_0 + \int\limits_{-\infty}^{\infty} f^{\psi}_{\beta}(x+t)\,\frac{1}{\pi}\int\limits_{0}^{\infty}\psi(v)\left[1+\frac{v}{2}\left(1-e^{-\frac{2}{\sigma}}\right)\right]e^{-\frac{v}{\sigma}}\cos\left(vt+\frac{\beta\pi}{2}\right)dvdt. \tag{2}$$

An operator B_σ, $\sigma \in (0,\infty)$, that acts on a function f according to rule (2) is called a Poisson biharmonic operator. Repeating the arguments used in the proof of Proposition 1.1 in [3, p. 169], we can easily verify that, under the condition of periodicity of f, the operator B_σ is the well-known Poisson biharmonic integral (see, e.g., [6]).

In the present paper, we study the asymptotic behavior $\big($as $\sigma \to \infty\big)$ of the quantity

$$\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}, B_\sigma\right)_{\hat{C}} = \sup_{f \in \hat{C}^{\psi}_{\beta,\infty}} \|f(x) - B_\sigma(f,x)\|_{\hat{C}} \tag{3}$$

for arbitrary real β and $\psi \in \mathfrak{A}$.

The investigation of structural and asymptotic properties of the classes $\hat{L}^{\psi}_{\beta}\mathfrak{N}$ was begun by Stepanets' [1, 2] and continued by his disciples. In particular, asymptotic equalities for the upper bounds of approximations of functions from the classes $\hat{C}^{\psi}_{\beta,\infty}$ and $\hat{L}^{\psi}_{\beta,1}$ by different linear operators were obtained by Dzimistarishvili [7–9], Rukasov and Chaichenko [10, 11], Ostrovs'ka [12], Repeta [13], Stepanets' and Sokolenko [14], Kal'chuk [15], etc.

In the present paper, we continue our investigation begun in [16]. In particular, we consider here the case where the function $\psi(v)$ that defines the class $\hat{C}^{\psi}_{\beta,\infty}$ tends to zero as $v \to \infty$ faster than the function $\frac{1}{v^2}$, which defines the order of the saturation of the linear approximation method generated by the operator B_σ.

We set

$$\tau(v) = \tau_\sigma(v;\psi) = \left(1-[1+\gamma v]\,e^{-v}\right)\frac{\psi(\sigma v)}{\psi(\sigma)}, \tag{4}$$

where the function $\psi \in \mathfrak{A}$ is defined and continuous for all $v \geq 0$ and

$$\gamma = \gamma_\sigma = \frac{\sigma}{2}\left(1-e^{-\frac{2}{\sigma}}\right).$$

Taking into account relation (4) and using (1) and (2), we obtain

$$f(x)-B_\sigma\left(f;x\right)=\psi(\sigma)\int\limits_{-\infty}^{\infty}f_\beta^\psi\left(x+\frac{t}{\sigma}\right)\hat{\tau}_\beta(t)dt, \tag{5}$$

where $\hat{\tau}_\beta(t)$ is the transform of the function $\tau(v)$ defined as follows:

$$\hat{\tau}_\beta(t)=\frac{1}{\pi}\int\limits_0^\infty\tau(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv.$$

We represent the function $\tau\left(v\right)$ in the form $\tau(v)=\varphi(v)+\mu(v),$ where

$$\varphi(v)=\left(\frac{v^2}{2}+\frac{v}{\sigma}\right)\frac{\psi(\sigma v)}{\psi(\sigma)},\quad v\geq 0, \tag{6}$$

$$\mu(v)=\left(1-\left[1+\gamma_\sigma v\right]e^{-v}-\frac{v^2}{2}-\frac{v}{\sigma}\right)\frac{\psi(\sigma v)}{\psi(\sigma)},\quad v\geq 0; \tag{7}$$

here, the function $\psi(\sigma v)$ is convex downward and increasing on the segment $\left[0,\frac{1}{\sigma}\right]$ and $\psi(0)=0.$

Further, let $f_1(x)$ and $f_2(x)$ be defined as follows:

$$f_1(x)=\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}f_\beta^\psi\left(x+t\right)\int\limits_0^\infty v\psi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dvdt, \tag{8}$$

$$f_2(x)=\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}f_\beta^\psi\left(x+t\right)\int\limits_0^\infty v^2\psi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dvdt, \tag{9}$$

where the function $\psi(v)$ is defined and continuous on the interval $[0,\infty)$ and $\beta\in R.$ The following statement is true:

Theorem 1. *If* $\psi\in\mathfrak{A}_C,$ *the function* $g(v)=v^2\psi(v)$ *is convex downward for* $v\in[b,\infty),$ $b\geq 1,$ *and*

$$\int\limits_1^\infty\frac{g(v)}{v}dv<\infty, \tag{10}$$

then the following asymptotic equality holds as $\sigma\to\infty$ *:*

$$\mathcal{E}\left(\hat{C}^\psi_{\beta,\infty};B_\sigma\right)_{\hat{C}}=\frac{1}{\sigma^2}\sup_{f\in\hat{C}^\psi_{\beta,\infty}}\left\|f_1(x)+\frac{f_2(x)}{2}\right\|_{\hat{C}}+O\left(\frac{1}{\sigma^3}\int\limits_1^\sigma t^2\psi(t)dt+\frac{1}{\sigma^2}\int\limits_\sigma^\infty t\psi(t)dt\right). \tag{11}$$

Proof. Let $\hat{\varphi}_\beta(t)$ and $\hat{\mu}_\beta(t)$ be the following transforms of the functions φ and μ:

$$\hat{\varphi}_\beta(t) = \frac{1}{\pi}\int\limits_0^{\infty} \varphi(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv, \tag{12}$$

$$\hat{\mu}_\beta(t) = \frac{1}{\pi}\int\limits_0^{\infty} \mu(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv. \tag{13}$$

Using the integral representation (5), we rewrite (3) in the form

$$\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}; B_\sigma\right)_{\hat{C}} = \sup_{f\in\hat{C}^{\psi}_{\beta,\infty}} \left\| \psi(\sigma)\int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\sigma}\right)\hat{\tau}_\beta(t)dt \right\|_{\hat{C}}$$

$$= \sup_{f\in\hat{C}^{\psi}_{\beta,\infty}} \left\| \psi(\sigma)\int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\sigma}\right)\left(\hat{\varphi}_\beta(t) + \hat{\mu}_\beta(t)\right) dt \right\|_{\hat{C}}. \tag{14}$$

Let us verify that the transforms $\hat{\varphi}_\beta(t)$ and $\hat{\mu}_\beta(t)$ defined by (12) and (13), respectively, are summable on the entire number axis.

First, we show the convergence of the integral $A(\varphi)$ defined as follows:

$$A(\varphi) = \int\limits_{-\infty}^{\infty} |\hat{\varphi}(t)|\, dt. \tag{15}$$

To this end, according to Theorem 1 in [17], it suffices to show the convergence of the integrals

$$\int\limits_0^{1/2} v|d\varphi'(v)|, \qquad \int\limits_{1/2}^{\infty} |v-1||d\varphi'(v)|, \tag{16}$$

$$\left|\sin\frac{\beta\pi}{2}\right| \int\limits_0^{\infty} \frac{|\varphi(v)|}{v}dv, \qquad \int\limits_0^{1} \frac{|\varphi(1-v) - \varphi(1+v)|}{v}dv. \tag{17}$$

Consider the first integral in (16). Using relation (6), we obtain

$$d\varphi'(v) = \frac{1}{\psi(\sigma)}\left(\psi(\sigma v) + 2\left(v + \frac{1}{\sigma}\right)\sigma\psi'(\sigma v) + \left(\frac{v^2}{2} + \frac{v}{\sigma}\right)\sigma^2\psi''(\sigma v)\right) dv. \tag{18}$$

Since the positive function $\psi(\sigma v)$ is convex downward and monotonically increasing on the segment $\left[0, \dfrac{1}{\sigma}\right]$, using (18) we get

$$d\varphi'(v) > 0, \quad v \in \left[0, \frac{1}{\sigma}\right]. \tag{19}$$

Taking into account that

$$\varphi\left(\frac{1}{\sigma}\right) = \frac{3\psi(1)}{2\sigma^2\psi(\sigma)} \qquad \text{and} \qquad \varphi'\left(\frac{1}{\sigma}\right) = \frac{4\psi(1) + 3\psi'(1-0)}{2\sigma\psi(\sigma)},$$

for $0 \le v \le \dfrac{1}{\sigma}$ we obtain

$$\int\limits_0^{1/\sigma} v|d\varphi'(v)| = \int\limits_0^{1/\sigma} v d\varphi'(v) = \frac{1}{\sigma}\varphi'\left(\frac{1}{\sigma}\right) - \varphi\left(\frac{1}{\sigma}\right) = O\left(\frac{1}{\sigma^2\psi(\sigma)}\right). \tag{20}$$

Taking into account that

$$\int\limits_{1/\sigma}^{1/2} v|d\varphi'(v)| \le \int\limits_{1/\sigma}^{\infty} v|d\varphi'(v)| \qquad \text{and} \qquad \int\limits_{1/2}^{\infty} |v-1||d\varphi'(v)| \le \int\limits_{1/\sigma}^{\infty} v|d\varphi'(v)|,$$

we estimate the integral

$$\int\limits_{1/\sigma}^{\infty} v|d\varphi'(v)| \tag{21}$$

on both intervals $\left[\dfrac{1}{\sigma}, \dfrac{b}{\sigma}\right)$ and $\left[\dfrac{b}{\sigma}, \infty\right)$ (for $\sigma > 2b$).

Using (18) and taking into account that the function $\psi(v)$ is convex downward for $v \ge 1$, we obtain

$$\int\limits_{1/\sigma}^{b/\sigma} v|d\varphi'(v)| \le \frac{1}{\psi(\sigma)} \int\limits_{1/\sigma}^{b/\sigma} \left(\frac{v^3}{2} + \frac{v^2}{\sigma}\right) \sigma^2 \psi''(\sigma v) dv + \frac{2}{\psi(\sigma)} \int\limits_{1/\sigma}^{b/\sigma} \left(v^2 + \frac{v}{\sigma}\right) \sigma |\psi'(\sigma v)| dv$$

$$+ \frac{1}{\psi(\sigma)} \int\limits_{1/\sigma}^{b/\sigma} v\psi(\sigma v) dv. \tag{22}$$

Integrating the first and the second integral on the right-hand side of (22) by parts and taking into account that $\psi(\sigma v) \leq \psi(1)$ for $v \in \left[\frac{1}{\sigma}, \frac{b}{\sigma}\right)$, we get

$$\int\limits_{1/\sigma}^{b/\sigma} v|d\varphi'(v)| \leq \frac{K_1}{\sigma^2\psi(\sigma)}. \tag{23}$$

To estimate integral (21) on the interval $\left[\frac{b}{\sigma}, \infty\right)$, we use the relations

$$\lim_{v\to\infty} v^2\psi(v) = 0, \tag{24}$$

$$\lim_{v\to\infty} v^3\psi'(v) = 0, \tag{25}$$

which are established by the following reasoning: Since the function $g(v) = v^2\psi(v)$ is convex downward for $v \geq b,\ b \geq 1,$ the following cases are possible:

(i) $\lim\limits_{v\to\infty} v^2\psi(v) = 0;$

(ii) $\lim\limits_{v\to\infty} v^2\psi(v) = K > 0;$

(iii) $\lim\limits_{v\to\infty} v^2\psi(v) = \infty.$

Let

$$\lim_{v\to\infty} v^2\psi(v) = K > 0.$$

Then there exists $0 < K_1 < K$ such that $v^2\psi(v) > K_1$ for all $v \geq 1,$ whence

$$v\psi(v) > \frac{K_1}{v},$$

which contradicts the condition of the theorem according to which the function $v\psi(v)$ is summable on $[1, \infty)$.

Now let

$$\lim_{v\to\infty} v^2\psi(v) = \infty,$$

i.e., for any $M > 0,$ one can find $N > 0$ such that $v^2\psi(v) > M$ for all $v > N.$ Then

$$\int\limits_1^x v\psi(v)dv = \int\limits_1^N v\psi(v)dv + \int\limits_N^x \frac{v^2\psi(v)}{v}dv > K_2 + \int\limits_N^x \frac{M}{v}dv = K_2 + M(\ln x - \ln N).$$

Thus, we again obtain a result that contradicts condition (10). Therefore, relation (24) is true.

We now show that relation (25) is true. Since the function $\left(v^2\psi(v)\right)'$ is summable on $[1,\infty)$, we have

$$\lim_{v\to\infty}\int\limits_{v/2}^{v}\left(x^2\psi(x)\right)'dx=0.$$

Since the function $v^2\psi(v)$ is convex downward for $v\ge b$, we conclude that the function $-\left(v^2\psi(v)\right)'$ does not increase for $v\ge b$ and, therefore,

$$\int\limits_{v/2}^{v}\left(-\left(x^2\psi(x)\right)'\right)dx>-\left(v-\frac{v}{2}\right)\left(2v\psi(v)+v^2\psi'(v)\right)=-v^2\psi(v)-\frac{1}{2}v^3\psi'(v).$$

Using this result and (24), we obtain relation (25).

Using (18) and taking into account the properties of the function $\psi(v)\in\mathfrak{M}$, $v\ge 1$, we get

$$\int\limits_{b/\sigma}^{\infty}v\left|d\varphi'(v)\right|\le\frac{1}{\psi(\sigma)}\int\limits_{b/\sigma}^{\infty}\left(\frac{v^3}{2}+\frac{v^2}{\sigma}\right)\sigma^2\psi''(\sigma v)dv+\frac{2}{\psi(\sigma)}\int\limits_{b/\sigma}^{\infty}\left(v^2+\frac{v}{\sigma}\right)\sigma\left|\psi'(\sigma v)\right|dv$$

$$+\frac{1}{\psi(\sigma)}\int\limits_{b/\sigma}^{\infty}v\psi(\sigma v)dv. \tag{26}$$

Integrating the first and the second integral on the right-hand side of inequality (26) by parts and taking into account relations (24), (25), and (10), we obtain

$$\int\limits_{b/\sigma}^{\infty}v|d\varphi'(v)|\le\frac{K_2}{\sigma^2\psi(\sigma)}. \tag{27}$$

Thus, it follows from (20), (23), and (27) that, as $\sigma\to\infty$, we have

$$\int\limits_{0}^{1/2}v|d\varphi'(v)|=O\left(\frac{1}{\sigma^2\psi(\sigma)}\right),\qquad\int\limits_{1/2}^{\infty}|v-1|\left|d\varphi'(v)\right|=O\left(\frac{1}{\sigma^2\psi(\sigma)}\right). \tag{28}$$

Using relation (6) and condition (10), we obtain the following estimate for the first integral in (17):

$$\int\limits_{0}^{\infty}\frac{|\varphi(v)|}{v}dv\le\frac{\psi(1)}{\psi(\sigma)}\int\limits_{0}^{1/\sigma}\left(\frac{v}{2}+\frac{1}{\sigma}\right)dv+\frac{1}{\psi(\sigma)}\int\limits_{1/\sigma}^{\infty}\left(\frac{v}{2}+\frac{1}{\sigma}\right)\psi(\sigma v)dv\le\frac{K}{\sigma^2\psi(\sigma)}.$$

Let us show that the second integral in (17) satisfies the following estimate as $\sigma \to \infty$:

$$\int_0^1 \frac{|\varphi(1-v) - \varphi(1+v)|}{v} dv = O\left(\frac{1}{\sigma^2 \psi(\sigma)}\right). \tag{29}$$

To obtain estimate (29), we use the following auxiliary statements:

Definition 1 [17]. *Assume that a function $\tau(v)$ is defined on $[0,\infty)$, absolutely continuous, and such that $\tau(\infty) = 0$. We say that the function $\tau(v)$ belongs to $\mathcal{E}_a$ if the derivative $\tau'(v)$ can be extended to the points where it does not exist so that, for a certain $a \geq 0$, the following integrals exist:*

$$\int_0^{a/2} v\big|d\tau'(v)\big| \qquad \textit{and} \qquad \int_{a/2}^{\infty} \big|v-a\big|\big|d\tau'(v)\big|.$$

Proposition 1 [17]. *If $\tau(v)$ belongs to $\mathcal{E}_a$, then*

$$\big|\tau(v)\big| \leq H(\tau),$$

where

$$H(\tau) = |\tau(0)| + |\tau(a)| + \int_0^{a/2} v|d\tau'(v)| + \int_{a/2}^{\infty} |v-a||d\tau'(v)|. \tag{30}$$

We set

$$\tau(v) = \tau_\sigma(v) = \big(1 - \lambda_\sigma(v)\big)\frac{\psi(\sigma v)}{\psi(\sigma)}, \quad \sigma \geq 1, \tag{31}$$

where the function ψ is defined and continuous for all $v \geq 0$.

Lemma 1. *Suppose that $\tau(v) \in \mathcal{E}_1$ and $\psi \in \mathfrak{A}_C$. Then the following relation holds as $\sigma \to \infty$:*

$$\int_0^1 \frac{|\tau(1-v) - \tau(1+v)|}{v} dv = O\left(\int_0^1 \frac{|\lambda_\sigma(1-v) - \lambda_\sigma(1+v)|}{v} dv + H(\tau)\right), \tag{32}$$

where $H(\tau)$ has the form (30).

Proof. Using relation (31), we determine the functions $\tau(1-v)$ and $\tau(1+v)$:

$$\tau(1-v) = (1 - \lambda_\sigma(1-v))\,\frac{\psi(\sigma(1-v))}{\psi(\sigma)}, \quad v \leq 1, \tag{33}$$

$$\tau(1+v) = (1 - \lambda_\sigma(1+v))\,\frac{\psi(\sigma(1+v))}{\psi(\sigma)}, \quad v \geq -1. \tag{34}$$

We represent the integral in (32) in the form of two integrals:

$$\int_0^1 \frac{|\tau(1-v)-\tau(1+v)|}{v}dv = \int_0^{1-1/\sigma} \frac{|\tau(1-v)-\tau(1+v)|}{v}dv + \int_{1-1/\sigma}^1 \frac{|\tau(1-v)-\tau(1+v)|}{v}dv. \qquad (35)$$

Let us estimate the first term on the right-hand side of (35). To this end, we add and subtract the following quantity under the modulus sign in the integrand:

$$\lambda_\sigma(1-v)-\lambda_\sigma(1+v).$$

As a result, we obtain

$$\int_0^{1-1/\sigma} \frac{|\tau(1-v)-\tau(1+v)|}{v}dv$$

$$\le \int_0^{1-1/\sigma} \frac{|\lambda_\sigma(1-v)-\lambda_\sigma(1+v)|}{v}dv + \int_0^{1-1/\sigma} \frac{|\tau(1-v)-\tau(1+v)+\lambda_\sigma(1-v)-\lambda_\sigma(1+v)|}{v}dv. \qquad (36)$$

Since, according to (33) and (34), one has

$$\lambda_\sigma(1-v) = 1 - \frac{\psi(\sigma)}{\psi(\sigma(1-v))}\tau(1-v) \qquad (37)$$

and

$$\lambda_\sigma(1+v) = 1 - \frac{\psi(\sigma)}{\psi(\sigma(1+v))}\tau(1+v), \qquad (38)$$

we obtain the following estimate for the second integral on the right-hand side of relation (36):

$$\int_0^{1-1/\sigma} \frac{|\tau(1-v)-\tau(1+v)+\lambda_\sigma(1-v)-\lambda_\sigma(1+v)|}{v}dv$$

$$\le \int_0^{1-1/\sigma} |\tau(1-v)| \left|1-\frac{\psi(\sigma)}{\psi(\sigma(1-v))}\right| \frac{dv}{v} + \int_0^{1-1/\sigma} |\tau(1+v)| \left|1-\frac{\psi(\sigma)}{\psi(\sigma(1+v))}\right| \frac{dv}{v}. \qquad (39)$$

Taking into account that $\tau(v)$ belongs to $\mathcal{E}_1$ and using Proposition 1, we get

$$\int\limits_0^{1-1/\sigma} |\tau(1-v)| \left|1 - \frac{\psi(\sigma)}{\psi(\sigma(1-v))}\right| \frac{dv}{v} + \int\limits_0^{1-1/\sigma} |\tau(1+v)| \left|1 - \frac{\psi(\sigma)}{\psi(\sigma(1+v))}\right| \frac{dv}{v}$$

$$= H(\tau) O \left(\int\limits_0^{1-1/\sigma} \frac{|\psi(\sigma(1-v)) - \psi(\sigma)|}{v\psi(\sigma(1-v))} dv + \int\limits_0^{1-1/\sigma} \frac{|\psi(\sigma(1+v)) - \psi(\sigma)|}{v\psi(\sigma(1+v))} dv \right). \quad (40)$$

Let us show that the following relations hold as $\sigma \to \infty$:

$$I_{1,\sigma} := \int\limits_0^{1-1/\sigma} \frac{|\psi(\sigma(1-v)) - \psi(\sigma)|}{v\psi(\sigma(1-v))} dv = O(1), \quad (41)$$

$$I_{2,\sigma} := \int\limits_0^{1-1/\sigma} \frac{|\psi(\sigma(1+v)) - \psi(\sigma)|}{v\psi(\sigma(1+v))} dv = O(1), \quad (42)$$

where $O(1)$ is uniformly bounded with respect to σ.

Further, we use the following statements:

Proposition 2 [5, p. 161]. *A function $\psi \in \mathfrak{M}$ belongs to $\mathfrak{M}_C$ if and only if the quantity*

$$\alpha(t) = \frac{\psi(t)}{t\,|\psi'(t)|}, \quad \psi'(t) := \psi'(t+0),$$

satisfies the condition

$$0 < K_1 \le \alpha(t) \le K_2 \quad \forall t \ge 1.$$

Proposition 3 [5, p. 175]. *For a function $\psi \in \mathfrak{M}$ to belong to $\mathfrak{M}_0$, it is necessary and sufficient that, for an arbitrary fixed number $c > 1$, there exist a constant K such that the following inequality holds for all $t \ge 1$:*

$$\frac{\psi(t)}{\psi(ct)} \le K.$$

Since the function

$$\frac{1 - \psi(\sigma)/\psi(\sigma(1-v))}{v}$$

is bounded for all $v \in \left[\delta, 1 - \frac{1}{\sigma}\right]$, $0 < \delta < 1 - \frac{1}{\sigma}$, taking into account Proposition 2 for $\psi \in \mathfrak{M}_0$ we get

$$\lim_{v\to 0} \frac{1 - \psi(\sigma)/\psi(\sigma(1-v))}{v} = \frac{\sigma\,|\psi'(\sigma)|}{\psi(\sigma)} \le K.$$

Thus, $I_{1,\sigma} = O(1)$ as $\sigma \to \infty$.

Passing to the estimation of the integral $I_{2,\sigma}$, we note that

$$I_{2,\sigma} < \frac{1}{\psi(2\sigma-1)} \int\limits_0^{1-1/\sigma} \frac{\psi(\sigma)-\psi(\sigma(1+v))}{v} dv.$$

Performing the change of variables $u=\sigma(1+v)$, we obtain

$$I_{2,\sigma} < \frac{1}{\psi(2\sigma-1)} \int\limits_\sigma^{2\sigma-1} \frac{\psi(\sigma)-\psi(u)}{u-\sigma} du < \frac{1}{\psi(2\sigma-1)} \int\limits_\sigma^{2\sigma} \frac{\psi(\sigma)-\psi(u)}{u-\sigma} du.$$

Applying Lemma 5.5 from [4, p. 97] to the right-hand side of the last inequality, taking into account that

$$\psi(2\sigma-1) \geq \psi(2\sigma), \quad \sigma \geq 1,$$

and using Proposition 3, we get

$$I_{2,\sigma} < \frac{K_1\psi(\sigma)}{\psi(2\sigma-1)} \leq \frac{K_1\psi(\sigma)}{\psi(2\sigma)} \leq K_2.$$

Combining relations (36) and (39)–(42), we write

$$\int\limits_0^{1-1/\sigma} \frac{|\tau(1-v)-\tau(1+v)|}{v} dv = \int\limits_0^{1-1/\sigma} \frac{|\lambda_\sigma(1-v)-\lambda_\sigma(1+v)|}{v} dv + O(1)H(\tau), \quad \sigma\to\infty. \tag{43}$$

Let us estimate the second term on the right-hand side of (35). To this end, we add and subtract the quantity

$$\frac{\psi(\sigma(1-v))}{\psi(1)} \left(\lambda_\sigma(1-v)-\lambda_\sigma(1+v)\right)$$

under the modulus sign in the integrand and take into account that the function $\psi(\sigma(1-v))$ is monotonically decreasing on $\left[1-\frac{1}{\sigma};1\right]$. As a result, we get

$$\int\limits_{1-1/\sigma}^{1} \frac{|\tau(1-v)-\tau(1+v)|}{v} dv$$

$$\leq \frac{1}{\psi(1)} \int\limits_{1-1/\sigma}^{1} \frac{\psi(\sigma(1-v))\,|\lambda_\sigma(1-v)-\lambda_\sigma(1+v)|}{v} dv$$

$$+ \int\limits_{1-1/\sigma}^{1} \frac{\left|\tau(1-v)-\tau(1+v)+\dfrac{\psi(\sigma(1-v))}{\psi(1)}\left(\lambda_\sigma(1-v)-\lambda_\sigma(1+v)\right)\right|}{v} dv$$

$$\leq \int\limits_{1-1/\sigma}^{1} \frac{|\lambda_\sigma(1-v)-\lambda_\sigma(1+v)|}{v}dv$$

$$+ \int\limits_{1-1/\sigma}^{1} \frac{\left|\tau(1-v)-\tau(1+v)+\frac{\psi\left(\sigma(1-v)\right)}{\psi(1)}\left(\lambda_\sigma(1-v)-\lambda_\sigma(1+v)\right)\right|}{v}dv. \quad (44)$$

Taking into account relations (37) and (38) and Proposition 1, we obtain

$$\int\limits_{1-1/\sigma}^{1} \frac{\left|\tau(1-v)-\tau(1+v)+\frac{\psi\left(\sigma(1-v)\right)}{\psi(1)}\left(\lambda_\sigma(1-v)-\lambda_\sigma(1+v)\right)\right|}{v}dv$$

$$\leq \int\limits_{1-1/\sigma}^{1} |\tau(1-v)|\left|1-\frac{\psi(\sigma)}{\psi(1)}\right|\frac{dv}{v} + \int\limits_{1-1/\sigma}^{1} |\tau(1+v)|\left|1-\frac{\psi\left(\sigma(1-v)\psi(\sigma)\right)}{\psi(1)\psi\left(\sigma(1+v)\right)}\right|\frac{dv}{v}$$

$$= H(\tau)O\left(\int\limits_{1-1/\sigma}^{1} \left|1-\frac{\psi(\sigma)}{\psi(1)}\right|\frac{dv}{v} + \int\limits_{1-1/\sigma}^{1} \left|1-\frac{\psi\left(\sigma(1-v)\right)\psi(\sigma)}{\psi(1)\psi\left(\sigma(1+v)\right)}\right|\frac{dv}{v}\right). \quad (45)$$

Further, we get

$$\int\limits_{1-1/\sigma}^{1} \left|1-\frac{\psi(\sigma)}{\psi(1)}\right|\frac{dv}{v} = \left(1-\frac{\psi(\sigma)}{\psi(1)}\right)\ln\frac{1}{1-\dfrac{1}{\sigma}} = O(1). \quad (46)$$

Since the function $\psi\left(\sigma(1-v)\right)$ is monotonically decreasing on the segment $\left[1-\dfrac{1}{\sigma};1\right]$, we conclude that $\psi(\sigma(1-v)) \leq \psi(1)$ and, furthermore, by virtue of Proposition 3 for $\sigma \geq 1$,

$$\frac{\psi(\sigma)}{\psi(\sigma(1+v))} \leq \frac{\psi(\sigma)}{\psi(2\sigma)} \leq K.$$

Therefore, the function $\left|1-\dfrac{\psi\left(\sigma(1-v)\right)\psi(\sigma)}{\psi(1)\psi\left(\sigma(1+v)\right)}\right|$ is bounded on $\left[1-\dfrac{1}{\sigma};1\right]$. Thus,

$$\int\limits_{1-1/\sigma}^{1} \left|1-\frac{\psi\left(\sigma(1-v)\right)\psi(\sigma)}{\psi(1)\psi\left(\sigma(1+v)\right)}\right|\frac{dv}{v} \leq K_1 \int\limits_{1-1/\sigma}^{1} \frac{dv}{v} = K\ln\frac{1}{1-\dfrac{1}{\sigma}} = O(1). \quad (47)$$

Using relations (45)–(47), we get

$$\int_{1-1/\sigma}^{1} \frac{\left|\tau(1-v)-\tau(1+v)+\dfrac{\psi\left(\sigma(1-v)\right)}{\psi(1)}\left(\lambda_\sigma(1-v)-\lambda_\sigma(1+v)\right)\right|}{v}dv = O\left(H(\tau)\right). \tag{48}$$

It follows from (44) and (48) that

$$\int_{1-1/\sigma}^{1} \frac{|\tau(1-v)-\tau(1+v)|}{v}dv = O\left(\int_{1-1/\sigma}^{1} \frac{|\lambda_\sigma(1-v)-\lambda_\sigma(1+v)|}{v}dv + H(\tau)\right). \tag{49}$$

Combining relations (43) and (49), we obtain equality (32).

The lemma is proved.

For the function φ defined by (6), we have

$$\lambda_\sigma(v) = \lambda_\sigma(\varphi; v) = 1 - \frac{\psi(\sigma)}{\psi(\sigma v)}\varphi(v) = 1 - \frac{v^2}{2} - \frac{v}{\sigma}.$$

It is easy to verify that

$$\int_{0}^{1} \frac{|\lambda_\sigma(1-v)-\lambda_\sigma(1+v)|}{v}dv = O\left(1\right), \quad \sigma\to\infty. \tag{50}$$

Using relations (30) and (6) and estimates (28) and taking into account relations (24), we get

$$H(\varphi) = O\left(1+\frac{1}{\sigma^2\psi(\sigma)}\right) = O\left(\frac{1}{\sigma^2\psi(\sigma)}\right), \quad \sigma\to\infty. \tag{51}$$

Combining relations (32), (50), and (51), we obtain estimate (29).

Thus, by virtue of Theorem 1 in [17], the integral $A(\varphi)$ given by (15) is convergent, and, hence, the transform $\hat{\varphi}_\beta(t)$ of the function φ defined by (6) is summable on the entire number axis.

The summability of the transform $\hat{\mu}_\beta(t)$ defined by (13) on the entire real axis follows from the convergence of the integral

$$A(\mu) = \int_{-\infty}^{\infty} |\hat{\mu}_\beta(t)|\, dt.$$

For the integral $A(\mu)$ to be convergent, it is necessary and sufficient (see Theorem 1 in [17, p. 24]) that the integrals

$$\int_{0}^{1/2} v|d\mu'(v)|, \qquad \int_{1/2}^{\infty} |v-1||d\mu'(v)|, \tag{52}$$

$$\left|\sin\frac{\beta\pi}{2}\right|\int\limits_0^\infty \frac{|\mu(v)|}{v}dv, \qquad \int\limits_0^1 \frac{|\mu(1-v)-\mu(1+v)|}{v}dv \tag{53}$$

be convergent. Let us estimate the first integral in (52) on each of the segments $\left[0,\frac{1}{\sigma}\right]$, $\left[\frac{1}{\sigma},\frac{b}{\sigma}\right]$, and $\left[\frac{b}{\sigma},\frac{1}{2}\right]$, $\sigma > 2b$. Denote

$$\overline{\mu}(v) = 1 - e^{-v} - \gamma v e^{-v} - \frac{v^2}{2} - \frac{v}{\sigma}. \tag{54}$$

By virtue of (7), the following equalities are true:

$$\mu(v) = \overline{\mu}(v)\frac{\psi(\sigma v)}{\psi(\sigma)}, \qquad \mu'(v) = \overline{\mu}'(v)\frac{\psi(\sigma v)}{\psi(\sigma)} + \overline{\mu}(v)\frac{\sigma\psi'(\sigma v)}{\psi(\sigma)}, \tag{55}$$

$$\mu''(v) = \overline{\mu}''(v)\frac{\psi(\sigma v)}{\psi(\sigma)} + 2\sigma\overline{\mu}'(v)\frac{\psi'(\sigma v)}{\psi(\sigma)} + \sigma^2\overline{\mu}(v)\frac{\psi''(\sigma v)}{\psi(\sigma)}. \tag{56}$$

Using (54), we obtain

$$\overline{\mu}'(v) = e^{-v} - \gamma e^{-v} + \gamma v e^{-v} - v - \frac{1}{\sigma},$$

$$\overline{\mu}''(v) = -e^{-v} + 2\gamma e^{-v} - \gamma v e^{-v} - 1,$$

$$\overline{\mu}(0) = 0, \qquad \overline{\mu}'(0) = 1 - \gamma - \frac{1}{\sigma} < 0.$$

These relations and the inequality

$$-1 + 2\gamma - \gamma v < e^v, \quad v \in [0,\infty),$$

yield

$$\overline{\mu}(v) \le 0, \qquad \overline{\mu}'(v) < 0, \qquad \overline{\mu}''(v) < 0 \qquad \text{for} \quad v \ge 0. \tag{57}$$

By virtue of inequalities (57) and the fact that the positive function $\psi(\sigma v)$ is convex downward and increasing on the segment $\left[0,\frac{1}{\sigma}\right]$, relation (56) yields

$$\mu''(v) < 0, \quad v \in \left[0,\frac{1}{\sigma}\right]. \tag{58}$$

We integrate the first integral in (52) by parts on the segment $\left[0, \frac{1}{\sigma}\right]$. Since $\mu(0) = 0$ and $\mu'(0) = 0$ $\big($because $\psi(0) = 0\big)$, taking into account inequality (58) and equalities (55) we get

$$\int\limits_0^{1/\sigma} v\left|d\mu'(v)\right| = -\int\limits_0^{1/\sigma} v d\mu'(v) = \overline{\mu}\left(\frac{1}{\sigma}\right)\frac{\psi(1)}{\psi(\sigma)} - \frac{1}{\sigma}\overline{\mu}'\left(\frac{1}{\sigma}\right)\frac{\psi(1)}{\psi(\sigma)} - \overline{\mu}\left(\frac{1}{\sigma}\right)\frac{\psi'(1-0)}{\psi(\sigma)}$$

$$\leq \frac{\psi(1)}{\sigma\psi(\sigma)}\left|\overline{\mu}'\left(\frac{1}{\sigma}\right)\right| + \frac{\psi'(1-0)}{\psi(\sigma)}\left|\overline{\mu}\left(\frac{1}{\sigma}\right)\right|. \tag{59}$$

Taking into account that

$$\left|\overline{\mu}(v)\right| < \frac{2}{3\sigma^2}v + \frac{1}{\sigma}v^2 + \frac{v^3}{2}, \qquad \left|\overline{\mu}'(v)\right| < \frac{2}{3\sigma^2} + \frac{2}{\sigma}v + \frac{3}{2}v^2, \quad v \geq 0, \tag{60}$$

and using relation (59), we obtain

$$\int\limits_0^{1/\sigma} v\left|d\mu'(v)\right| \leq \frac{K_1}{\sigma^3\psi(\sigma)}. \tag{61}$$

Let us estimate the first integral in (52) on the segment $\left[\frac{1}{\sigma}, \frac{b}{\sigma}\right]$. Using (56), we get

$$\int\limits_{1/\sigma}^{b/\sigma} v\left|d\mu'(v)\right| \leq \frac{1}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma} v|\overline{\mu}''(v)|\psi(\sigma v)dv + \frac{2\sigma}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma} v|\overline{\mu}'(v)||\psi'(\sigma v)|dv + \frac{\sigma^2}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma} v|\overline{\mu}(v)|\psi''(\sigma v)dv.$$

Taking into account inequalities (60) and the estimate

$$\left|\overline{\mu}''(v)\right| < \frac{2}{\sigma} + 3v, \quad v \geq 0,$$

and integrating by parts, we obtain

$$\int\limits_{1/\sigma}^{b/\sigma} v\left|d\mu'(v)\right| \leq \frac{K_2}{\sigma^3\psi(\sigma)}. \tag{62}$$

We show that if the function $v^2\psi(v)$ is convex downward for $v \geq b$, $b \geq 1$, then

$$d\mu'(v) \leq 0, \quad v \geq \frac{b}{\sigma}. \tag{63}$$

To this end, we set

$$\tilde{\mu}(v) = \frac{\overline{\mu}(v)}{v^2}.$$

According to (54), we have

$$\tilde{\mu}(v) = \frac{1}{v^2} - \frac{e^{-v}}{v^2} - \gamma\frac{e^{-v}}{v} - \frac{1}{2} - \frac{1}{v\sigma}.$$

Since

$$\tilde{\mu}'(v) = \frac{1}{v^3}\left(-2 + 2e^{-v} + (1+\gamma)ve^{-v} + \gamma v^2 e^{-v} + \frac{v}{\sigma}\right),$$

$$\tilde{\mu}''(v) = \frac{1}{v^4}\left(6 - 6e^{-v} - (4+2\gamma)ve^{-v} - (1+2\gamma)v^2 e^{-v} - \gamma v^3 e^{-v} - \frac{2v}{\sigma}\right),$$

taking into account the inequalities $e^{-v} \geq 1 - v,\ \ v \geq 0,$ and $\gamma > 1 - \dfrac{1}{\sigma}$ we obtain

$$\tilde{\mu}(v) < 0,$$

$$\tilde{\mu}'(v) > \frac{1}{v^3}\left(\frac{v^2}{\sigma} + \gamma v^2 e^{-v}\right) > 0,$$

$$\tilde{\mu}''(v) < \frac{1}{v^4}\left(-\frac{2v^2}{\sigma} - (1+2\gamma)v^2 e^{-v} - \gamma v^3 e^{-v}\right) < 0.$$

For $v \geq b \geq 1,$ according to the conditions of the theorem, the following relations are true:

$$g(v) > 0, \qquad g'(v) < 0, \qquad g''(v) > 0.$$

Then

$$\mu''(v) = \left(\frac{1}{\sigma^2}\tilde{\mu}(v)g(\sigma v)\right)'' = \frac{1}{\sigma^2}\tilde{\mu}''(v)g(\sigma v) + \frac{2}{\sigma}\tilde{\mu}'(v)g'(\sigma v) + \tilde{\mu}(v)g''(\sigma v) < 0 \quad \text{for} \quad v \geq \frac{b}{\sigma},$$

and, hence, inequality (63) holds for all $v \geq b/\sigma$ and $b \geq 1.$

Using inequality (63), relations (55) and (60), and Propositions 2 and 3, we get

$$\int\limits_{b/\sigma}^{1/2} v\left|d\mu'(v)\right| = -\int\limits_{b/\sigma}^{1/2} v d\mu'(v)$$

$$= -\frac{1}{2}\mu'\left(\frac{1}{2}\right) + \frac{b}{\sigma}\mu'\left(\frac{b}{\sigma}\right) + \mu\left(\frac{1}{2}\right) - \mu\left(\frac{b}{\sigma}\right) \leq K_1 + \frac{K_2}{\sigma^3\psi(\sigma)}, \quad \sigma \to \infty. \tag{64}$$

Combining relations (61), (62), and (64), we obtain the following estimate for the first integral in (52):

$$\int\limits_{0}^{1/2} v|d\mu'(v)| = O\left(1 + \frac{1}{\sigma^3\psi(\sigma)}\right), \quad \sigma \to \infty. \tag{65}$$

Taking into account relations (24) and (25) and Propositions 2 and 3, one can easily verify that the following estimate holds for the second integral in (52):

$$\int\limits_{1/2}^{\infty} |v-1||d\mu'(v)| = O(1), \quad \sigma \to \infty. \tag{66}$$

Let us estimate the first integral in (53) on each of the intervals $\left[0, \frac{1}{\sigma}\right]$, $\left[\frac{1}{\sigma}, 1\right]$, and $\left[\frac{1}{\sigma}, \infty\right)$. Since the function $\overline{\mu}(v)$ defined by (54) is nonpositive for $v \geq 0$, using the first relation in (55) we get

$$|\mu(v)| = -\overline{\mu}(v)\frac{\psi(\sigma v)}{\psi(\sigma)}.$$

Using the inequality

$$e^{-v} \leq 1 - v + \frac{v^2}{2}, \quad v \geq 0, \tag{67}$$

and the fact that the function $\psi(\sigma v)$ is increasing for $v \in \left[0, \frac{1}{\sigma}\right]$, we obtain

$$\int\limits_{0}^{1/\sigma} \frac{|\mu(v)|}{v} dv = \frac{1}{\psi(\sigma)} \int\limits_{0}^{1/\sigma} \left(-1 + e^{-v} + \gamma v e^{-v} + \frac{v^2}{2} + \frac{v}{\sigma}\right) \frac{\psi(\sigma v)}{v} dv$$

$$\leq \frac{\psi(1)}{\psi(\sigma)} \int\limits_{0}^{1/\sigma} \left(-1 + \gamma + \frac{1}{\sigma} + (1-\gamma)v + \frac{\gamma}{2}v^2\right) dv.$$

Using the last relation and the inequalities

$$-1 + \gamma + \frac{1}{\sigma} < \frac{2}{3\sigma^2}, \qquad \gamma < 1, \quad 1 - \gamma < \frac{1}{\sigma}, \tag{68}$$

we get

$$\int\limits_{0}^{1/\sigma} \frac{|\mu(v)|}{v} dv = O\left(\frac{1}{\sigma^3 \psi(\sigma)}\right), \quad \sigma \to \infty. \tag{69}$$

Taking into account inequalities (67) and (68), we obtain

$$\int\limits_{1/\sigma}^{1} \frac{|\mu(v)|}{v} dv \le \int\limits_{1/\sigma}^{1} \frac{\psi(\sigma v)}{\psi(\sigma)} \left(\frac{1}{\sigma} + \gamma - 1 + (1-\gamma)v + \frac{\gamma}{2!} v^2 \right) dv$$

$$\le \frac{K_1}{\sigma^3 \psi(\sigma)} \int\limits_{1}^{\sigma} \psi(v) dv + \frac{K_2}{\sigma^3 \psi(\sigma)} \int\limits_{1}^{\sigma} v\psi(v) dv + \frac{K_3}{\sigma^3 \psi(\sigma)} \int\limits_{1}^{\sigma} v^2 \psi(v) dv$$

$$= O\left(\frac{1}{\sigma^3 \psi(\sigma)} \int\limits_{1}^{\sigma} v^2 \psi(v) dv \right), \tag{70}$$

$$\int\limits_{1}^{\infty} \frac{|\mu(v)|}{v} dv = \frac{1}{\psi(\sigma)} \int\limits_{1}^{\infty} \psi(\sigma v) \left(\frac{e^{-v} - 1}{v} + \gamma e^{-v} + \frac{v}{2} + \frac{1}{\sigma} \right) dv$$

$$\le \frac{1}{\psi(\sigma)} \int\limits_{1}^{\infty} \psi(\sigma v) \left(-1 + \frac{v}{2} + \gamma + \frac{v}{2} + \frac{1}{\sigma} \right) dv$$

$$= O\left(\frac{1}{\sigma^2 \psi(\sigma)} \int\limits_{\sigma}^{\infty} v\psi(v) dv \right). \tag{71}$$

Combining relations (69)–(71) and taking into account that

$$\int\limits_{1}^{\sigma} v^2 \psi(v) dv \ge K,$$

we obtain the following estimate for the first integral in (53):

$$\int\limits_{0}^{\infty} \frac{|\mu(v)|}{v} dv = O\left(\frac{1}{\sigma^3 \psi(\sigma)} \int\limits_{1}^{\sigma} v^2 \psi(v) dv + \frac{1}{\sigma^2 \psi(\sigma)} \int\limits_{\sigma}^{\infty} v\psi(v) dv \right). \tag{72}$$

Let us estimate the second integral in (53). To this end, we use relation (32) for

$$\overline{\lambda}(v) = \lambda_\sigma(\mu; v) = 1 - \frac{\psi(\sigma)}{\psi(\sigma v)} \mu(v) = [1 + \gamma v]\, e^{-v} + \frac{v^2}{2} + \frac{v}{\sigma}.$$

It is easy to verify that

$$\int\limits_{0}^{1} \frac{\left|\overline{\lambda}(1-v) - \overline{\lambda}(1+v)\right|}{v} dv = \int\limits_{0}^{1} \left| \frac{\gamma+1}{e} \frac{e^v - e^{-v}}{v} - \frac{\gamma}{e}(e^v + e^{-v}) + 2\left(1 + \frac{1}{\sigma}\right) \right| dv = O(1), \quad \sigma \to \infty. \tag{73}$$

Furthermore, relations (30), (7), (65), and (66) yield the following estimate:

$$H(\mu)=O\left(1+\frac{1}{\sigma^3\psi(\sigma)}\right),\quad \sigma\to\infty. \tag{74}$$

Comparing (73) and (74) and using (32), we get

$$\int_0^1 \frac{|\mu(1-v)-\mu(1+v)|}{v}dv=O\left(1+\frac{1}{\sigma^3\psi(\sigma)}\right),\quad \sigma\to\infty. \tag{75}$$

Thus, the transform $\hat{\mu}_\beta(t)$ given by (13) for the function μ defined by (7) is summable on the entire real axis. Using (14), we obtain

$$\mathcal{E}\left(\hat{C}^\psi_{\beta,\infty};B_\sigma\right)_{\hat{C}}=\sup_{f\in\hat{C}^\psi_{\beta,\infty}}\left\|\psi(\sigma)\int_{-\infty}^{+\infty}f^\psi_\beta\left(x+\frac{t}{\sigma}\right)(\hat{\varphi}_\beta(t)+\hat{\mu}_\beta(t))\,dt\right\|_{\hat{C}}$$

$$=\sup_{f\in\hat{C}^\psi_{\beta,\infty}}\left\|\psi(\sigma)\int_{-\infty}^{+\infty}f^\psi_\beta\left(x+\frac{t}{\sigma}\right)\hat{\varphi}_\beta(t)dt\right\|_{\hat{C}}+O\left(\psi(\sigma)A(\mu)\right). \tag{76}$$

Taking into account relations (6), (8), and (9), we get

$$\int_{-\infty}^{+\infty}f^\psi_\beta\left(x+\frac{t}{\sigma}\right)\hat{\varphi}_\beta(t)dt=\frac{1}{\sigma^2\psi(\sigma)}\left(f_1(x)+\frac{f_2(x)}{2}\right). \tag{77}$$

Using (76) and (77), we obtain

$$\mathcal{E}\left(\hat{C}^\psi_{\beta,\infty};B_\sigma\right)_{\hat{C}}=\frac{1}{\sigma^2}\sup_{f\in\hat{C}^\psi_{\beta,\infty}}\left\|f_1(x)+\frac{f_2(x)}{2}\right\|_{\hat{C}}+O\left(\psi(\sigma)A(\mu)\right). \tag{78}$$

Furthermore, according to formulas (2.14) and (2.15) from [17, p. 25] and relations (72), (74), and (75), the following estimate holds for $A(\mu)$:

$$A(\mu)=O\left(1+\frac{1}{\sigma^3\psi(\sigma)}+\frac{1}{\sigma^3\psi(\sigma)}\int_1^\sigma v^2\psi(v)dv+\frac{1}{\sigma^2\psi(\sigma)}\int_\sigma^\infty v\psi(v)dv\right).$$

Taking into account that

$$\int_1^\sigma v^2\psi(v)dv\geq K \qquad \text{and} \qquad \frac{1}{\sigma^3\psi(\sigma)}\int_1^\sigma v^2\psi(v)dv\geq K,$$

we conclude that

$$A(\mu) = O\left(\frac{1}{\sigma^3\psi(\sigma)}\int\limits_1^{\sigma} v^2\psi(v)dv + \frac{1}{\sigma^2\psi(\sigma)}\int\limits_{\sigma}^{\infty} v\psi(v)dv\right). \tag{79}$$

Using (78) and (79), we obtain relation (11).

Theorem 1 is proved.

Note that the conditions of Theorem 1 are satisfied, e.g., by the functions $\psi \in \mathfrak{A}$ that, for $v \geq 1,$ have the forms

$$\psi(v) = \frac{1}{v^2}\ln^{\alpha}(v+K),$$

where

$$K > 0 \quad \text{and} \quad \alpha < -1,$$

and

$$\psi(v) = \frac{1}{v^r}(K + e^{-v}), \quad \psi(v) = \frac{1}{v^r}\ln^{\alpha}(v+K), \quad \text{and} \quad \psi(v) = \frac{1}{v^r}\arctan v,$$

where

$$K > 0, \quad r > 2, \quad \text{and} \quad \alpha \in R.$$

Theorem 2. *If* $\psi \in \mathfrak{A},$ *the function* $g(v) = v^2\psi(v)$ *is convex downward for* $v \geq b \geq 1,$ *and*

$$\int\limits_1^{\infty} vg(v)dv < \infty,$$

then the following asymptotic equality holds as $\sigma \to \infty:$

$$\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}; B_{\sigma}\right)_{\hat{C}} = \frac{1}{\sigma^2}\sup_{f\in\hat{C}^{\psi}_{\beta,\infty}}\left\|f_1(x) + \frac{f_2(x)}{2}\right\|_{\hat{C}} + O\left(\frac{1}{\sigma^3}\right), \tag{80}$$

where the functions $f_1(x)$ *and* $f_2(x)$ *are defined by (8) and (9), respectively.*

Proof. Let $\tau(v) = \varphi(v) + \mu(v),$ where the functions $\varphi(v)$ and $\mu(v)$ are defined by (6) and (7), respectively. Taking (3) and (5) into account, we obtain relation (14):

$$\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}; B_{\sigma}\right)_{\hat{C}} = \sup_{f\in\hat{C}^{\psi}_{\beta,\infty}}\left\|\psi(\sigma)\int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\sigma}\right)(\hat{\varphi}_{\beta}(t) + \hat{\mu}_{\beta}(t))\,dt\right\|_{\hat{C}},$$

where $\hat{\varphi}_{\beta}(t)$ and $\hat{\mu}_{\beta}(t)$ are transforms (12) and (13) for the functions φ and $\mu,$ respectively.

Let us show that the transforms $\hat{\varphi}_\beta(t)$ and $\hat{\mu}_\beta(t)$ are summable on the entire real axis. First, we prove the convergence of the integral $A(\varphi)$ defined by (15). To this end, we divide the interval $(-\infty,+\infty)$ into two subsets $(-\sigma,\sigma)$ and $(-\infty,\sigma]\cup[\sigma,+\infty)$.

Let us estimate the integral $A(\varphi)$ given by (15) on the interval $(-\sigma,\sigma)$. Using relation (6) and the fact that the function $\psi(\sigma v)$ increases for $v\in\left[0,\dfrac{1}{\sigma}\right]$ and taking into account that

$$\int\limits_1^\infty vg(v)dv<\infty,$$

we get

$$\int\limits_{-\sigma}^{\sigma}\left|\int\limits_0^\infty \varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt$$

$$\le 2\sigma\int\limits_0^\infty|\varphi(v)|dv=\frac{2\sigma}{\psi(\sigma)}\int\limits_0^\infty\left(\frac{v^2}{2}+\frac{v}{\sigma}\right)\psi(\sigma v)dv$$

$$\le\frac{2\sigma\psi(1)}{\psi(\sigma)}\int\limits_0^{1/\sigma}\left(\frac{v^2}{2}+\frac{v}{\sigma}\right)dv+\frac{2\sigma}{\psi(\sigma)}\int\limits_{1/\sigma}^{\infty}\left(\frac{v^2}{2}+\frac{v}{\sigma}\right)\psi(\sigma v)dv\le\frac{K_1}{\sigma^2\psi(\sigma)}. \tag{81}$$

Let us estimate integral (15) for $|t|\ge\sigma$. To this end, we consider the integral

$$\int\limits_0^\infty \varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv$$

on the intervals $\left[0;\dfrac{1}{\sigma}\right]$ and $\left[\dfrac{1}{\sigma};\infty\right]$:

$$\int\limits_0^\infty \varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv=\left(\int\limits_0^{1/\sigma}+\int\limits_{1/\sigma}^{\infty}\right)\varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv. \tag{82}$$

According to (6),

$$\varphi(0)=0,\quad \varphi\left(\frac{1}{\sigma}\right)=\frac{3\psi(1)}{2\sigma^2\psi(\sigma)},$$

and, for $v\in\left[0,\dfrac{1}{\sigma}\right)$,

$$\varphi'(0)=0,\qquad \varphi'\left(\frac{1}{\sigma}\right)=\frac{4\psi(1)+3\psi'(1-0)}{2\sigma\psi(\sigma)}. \tag{83}$$

Integrating the first integral on the right-hand side of (82) twice by parts, we obtain

$$\int_0^{1/\sigma} \varphi(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv$$

$$= \frac{3\psi(1)}{2t\sigma^2\psi(\sigma)} \sin\left(\frac{t}{\sigma} + \frac{\beta\pi}{2}\right) + \frac{4\psi(1) + 3\psi'(1-0)}{2t^2\sigma\psi(\sigma)} \cos\left(\frac{t}{\sigma} + \frac{\beta\pi}{2}\right)$$

$$- \frac{1}{t^2} \int_0^{1/\sigma} \varphi''(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv. \tag{84}$$

Further, since the function $g(v) = v^2\psi(v)$ is convex downward and

$$\int_1^{\infty} vg(v)dv < \infty,$$

relations (24) and (25) are true. Integrating the second integral on the right-hand side of (82) twice by parts on the interval $\left[\frac{1}{\sigma}, \infty\right)$ and taking into account that

$$\lim_{v\to\infty} \varphi(v) = \lim_{v\to\infty} \varphi'(v) = 0,$$

we get

$$\int_{1/\sigma}^{\infty} \varphi(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv = -\varphi\left(\frac{1}{\sigma}\right) \sin\left(\frac{t}{\sigma} + \frac{\beta\pi}{2}\right) - \frac{1}{t} \int_{1/\sigma}^{\infty} \varphi'(v)\sin\left(vt + \frac{\beta\pi}{2}\right) dv$$

$$= -\frac{3\psi(1)}{2t\sigma^2\psi(\sigma)} \sin\left(\frac{t}{\sigma} + \frac{\beta\pi}{2}\right) - \frac{1}{t^2} \frac{4\psi(1) + 3\psi'(1)}{2\sigma\psi(\sigma)} \cos\left(\frac{t}{\sigma} + \frac{\beta\pi}{2}\right)$$

$$- \frac{1}{t^2} \int_{1/\sigma}^{\infty} \varphi''(v)\cos\left(vt + \frac{\beta\pi}{2}\right) dv. \tag{85}$$

Combining relations (82)–(85), we write

$$\int\limits_0^\infty \varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv$$

$$=\frac{3\psi'(1-0)-3\psi'(1)}{2t^2\sigma\psi(\sigma)}\cos\left(\frac{t}{\sigma}+\frac{\beta\pi}{2}\right)-\frac{1}{t^2}\int\limits_0^{1/\sigma}\varphi''(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv$$

$$-\frac{1}{t^2}\int\limits_{1/\sigma}^{\infty}\varphi''(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv.$$

This yields

$$\left|\int\limits_0^\infty \varphi(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|\le\frac{3\psi'(1-0)-3\psi'(1)}{2t^2\sigma\psi(\sigma)}+\frac{1}{t^2}\int\limits_0^{1/\sigma}|\varphi''(v)|dv+\frac{1}{t^2}\int\limits_{1/\sigma}^{\infty}|\varphi''(v)|dv. \tag{86}$$

Taking relations (19) and (83) into account, we obtain

$$\int\limits_0^{1/\sigma}|\varphi''(v)|dv=\varphi'\left(\frac{1}{\sigma}\right)-\varphi'(0)=\frac{K}{\sigma\psi(\sigma)}. \tag{87}$$

Further, using relation (18) and the fact that the function $\psi(\sigma v)$, $v\in\left[\frac{1}{\sigma},\infty\right)$, decreases and is convex downward, we get

$$\int\limits_{1/\sigma}^{b/\sigma}|\varphi''(v)|dv\le\frac{1}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma}\psi(\sigma v)dv+\frac{2\sigma}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma}\left(v+\frac{1}{\sigma}\right)\left|\psi'(\sigma v)\right|dv+\frac{\sigma^2}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma}\left(\frac{v^2}{2}+\frac{v}{\sigma}\right)\psi''(\sigma v)dv. \tag{88}$$

It is easy to verify that

$$\frac{\sigma^2}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma}\left(\frac{v^2}{2}+\frac{v}{\sigma}\right)\psi''(\sigma v)dv=\frac{K_1}{\sigma\psi(\sigma)}-\frac{\sigma}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma}\left(v+\frac{1}{\sigma}\right)\psi'(\sigma v)dv.$$

Combining the last relation with inequality (88) and taking into account that

$$\frac{1}{\psi(\sigma)}\int\limits_{1/\sigma}^{b/\sigma}\psi(\sigma v)dv\le\frac{(b-1)\psi(1)}{\sigma\psi(\sigma)},$$

we obtain

$$\int_{1/\sigma}^{b/\sigma} |\varphi''(v)|dv \leq \frac{K_2}{\sigma\psi(\sigma)} + \frac{3\sigma}{\psi(\sigma)} \int_{1/\sigma}^{b/\sigma} \left(v + \frac{1}{\sigma}\right) \left|\psi'(\sigma v)\right| dv.$$

Integrating the integral on the right-hand side of the last inequality by parts, we get

$$\int_{1/\sigma}^{b/\sigma} |\varphi''(v)|dv \leq \frac{K}{\sigma\psi(\sigma)}. \tag{89}$$

Using relations (18), (24), and (25) and taking into account that $\psi(v)$ is decreasing for $v \in [1, \infty)$ and

$$\lim_{v\to\infty} \psi(v) = 0,$$

we obtain the following estimate:

$$\frac{1}{t^2} \int_{1/\sigma}^{\infty} |\varphi''(v)|dv \leq \frac{K}{t^2\sigma\psi(\sigma)}.$$

Hence, using relations (86)–(89), we get

$$\left| \int_{0}^{\infty} \varphi(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| \leq \frac{K}{t^2\sigma\psi(\sigma)}.$$

Therefore,

$$\int_{|t|\geq\sigma} |\hat{\varphi}(t)|\, dt \leq \frac{2K}{\sigma^2\psi(\sigma)}. \tag{90}$$

Using relations (81) and (90), we obtain the following estimate for the integral $A(\varphi)$ defined by (15):

$$A(\varphi) = \frac{O(1)}{\sigma^2\psi(\sigma)}.$$

Thus, the transform $\hat{\varphi}_\beta(t)$ defined by (12) is summable on the entire number axis.

Further, we verify the summability of the integral

$$A(\mu) = \int_{-\infty}^{\infty} \left|\hat{\mu}_\beta(t)\right| dt,$$

where $\hat{\mu}_\beta(t)$ is transform (13) for the function $\mu(v)$. To this end, we rewrite the integral $A(\mu)$ in the form

$$A(\mu) = \frac{1}{\pi}\int_{-\sigma}^{\sigma}\left|\int_0^\infty \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt + \frac{1}{\pi}\int_{|t|\geq\sigma}\left|\int_0^\infty \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt = I_1 + I_2. \qquad (91)$$

We estimate the integral I_1 as follows:

$$I_1 \leq \frac{1}{\pi}\int_{-\sigma}^{\sigma}\left|\int_0^{1/\sigma} \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt + \frac{1}{\pi}\int_{-\sigma}^{\sigma}\left|\int_{1/\sigma}^\infty \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt = I_3 + I_4. \qquad (92)$$

Using the first relations in (55) and (60) and the inequality $\psi(\sigma v) \leq \psi(1)$ for $v \in \left[0, \frac{1}{\sigma}\right]$, we obtain the following estimate for the integral I_3:

$$I_3 \leq \frac{1}{\pi}\int_{-\sigma}^{\sigma}\int_0^{1/\sigma}|\mu(v)|\,dvdt \leq \frac{2\sigma\psi(1)}{\pi\psi(\sigma)}\int_0^{1/\sigma}\left(\frac{2v}{3\sigma^2}+\frac{v^2}{\sigma}+\frac{v^3}{2}\right)dv = \frac{K}{\sigma^3\psi(\sigma)}. \qquad (93)$$

According to the theorem, we have

$$\int_1^\infty v^3\psi(v)dv < \infty.$$

Using again the first inequality from (60), we obtain the following estimate for the integral I_4:

$$I_4 \leq \frac{1}{\pi}\int_{-\sigma}^{\sigma}\int_{1/\sigma}^{\infty}|\mu(v)|dvdt$$

$$= \frac{2\sigma}{\pi\psi(\sigma)}\left(\frac{2}{3\sigma^4}\int_1^\infty v\psi(v)dv + \frac{1}{\sigma^4}\int_1^\infty v^2\psi(v)dv + \frac{1}{2\sigma^4}\int_1^\infty v^3\psi(v)dv\right) \leq \frac{K}{\sigma^3\psi(\sigma)}. \qquad (94)$$

Combining relations (92)–(94), we write

$$I_1 = \frac{1}{\pi}\int_{-\sigma}^{\sigma}\left|\int_0^\infty \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|dt = \frac{O(1)}{\sigma^3\psi(\sigma)}, \quad \sigma\to\infty. \qquad (95)$$

Let us estimate the integral I_2. Integrating twice by parts and taking into account that $\mu(0) = 0$ and $\mu'(0) = 0$, we get

$$\int\limits_0^{1/\sigma} \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv$$

$$=\frac{1}{t}\mu\left(\frac{1}{\sigma}\right)\sin\left(\frac{t}{\sigma}+\frac{\beta\pi}{2}\right)+\frac{1}{t^2}\mu'\left(\frac{1}{\sigma}-0\right)\cos\left(\frac{t}{\sigma}+\frac{\beta\pi}{2}\right)-\frac{1}{t^2}\int\limits_0^{1/\sigma}\mu''(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv. \quad (96)$$

Taking relations (24) and (25) into account, we obtain

$$\lim_{v\to\infty}\mu(v)=0 \qquad \text{and} \qquad \lim_{v\to\infty}\mu'(v)=0.$$

Then

$$\int\limits_{1/\sigma}^{\infty} \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv$$

$$=-\frac{1}{t}\mu\left(\frac{1}{\sigma}\right)\sin\left(\frac{t}{\sigma}+\frac{\beta\pi}{2}\right)-\frac{1}{t^2}\mu'\left(\frac{1}{\sigma}\right)\cos\left(\frac{t}{\sigma}+\frac{\beta\pi}{2}\right)-\frac{1}{t^2}\int\limits_{1/\sigma}^{\infty}\mu''(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv. \quad (97)$$

Combining relations (96) and (97), we get

$$\int\limits_0^{\infty} \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv$$

$$=\frac{1}{t^2}\left(\mu'\left(\frac{1}{\sigma}-0\right)-\mu'\left(\frac{1}{\sigma}\right)\right)\cos\left(\frac{t}{\sigma}+\frac{\beta\pi}{2}\right)$$

$$-\frac{1}{t^2}\int\limits_0^{1/\sigma}\mu''(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv-\frac{1}{t^2}\int\limits_{1/\sigma}^{\infty}\mu''(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv.$$

It follows from the second relation in (55) that

$$\mu'\left(\frac{1}{\sigma}-0\right)=\overline{\mu}'\left(\frac{1}{\sigma}\right)\frac{\psi(1)}{\psi(\sigma)}+\overline{\mu}\left(\frac{1}{\sigma}\right)\frac{\sigma\psi'(1-0)}{\psi(\sigma)}, \quad (98)$$

$$\mu'\left(\frac{1}{\sigma}\right)=\overline{\mu}'\left(\frac{1}{\sigma}\right)\frac{\psi(1)}{\psi(\sigma)}+\overline{\mu}\left(\frac{1}{\sigma}\right)\frac{\sigma\psi'(1)}{\psi(\sigma)}. \quad (99)$$

Therefore,

$$\int_0^\infty \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv$$

$$=\frac{1}{t^2}\overline{\mu}\left(\frac{1}{\sigma}\right)\frac{\sigma\left(\psi'(1-0)-\psi'(1)\right)}{\psi(\sigma)}\cos\left(\frac{t}{\sigma}+\frac{\beta\pi}{2}\right)-\frac{1}{t^2}\left[\int_0^{1/\sigma}+\int_{1/\sigma}^\infty\right]\mu''(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv. \quad (100)$$

Using relation (100) and taking into account the first inequality in (60), we obtain

$$\left|\int_0^\infty \mu(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\right|\le\frac{K_1}{t^2\sigma^2\psi(\sigma)}+\frac{1}{t^2}\int_0^{1/\sigma}|\mu''(v)|dv+\frac{1}{t^2}\int_{1/\sigma}^\infty|\mu''(v)|dv. \quad (101)$$

Using relations (58) and (98) and the fact that $\mu'(0)=0,$ we get

$$\int_0^{1/\sigma}|\mu''(v)|dv=-\mu'\left(\frac{1}{\sigma}-0\right)=\left|\overline{\mu}'\left(\frac{1}{\sigma}\right)\right|\frac{\psi(1)}{\psi(\sigma)}+\left|\overline{\mu}\left(\frac{1}{\sigma}\right)\right|\frac{\sigma\psi'(1-0)}{\psi(\sigma)}.$$

Taking into account both relations in (60), we obtain

$$\int_0^{1/\sigma}|\mu''(v)|dv\le\frac{K_2}{\sigma^2\psi(\sigma)}. \quad (102)$$

Consider the second integral on the right-hand side of inequality (101) on each of the intervals $\left[\frac{1}{\sigma},\frac{b}{\sigma}\right]$ and $\left[\frac{b}{\sigma},\infty\right)$. Taking (56) into account and reasoning as in the proof of relation (62), we get

$$\int_{1/\sigma}^{b/\sigma}|\mu''(v)|dv\le\frac{K_3}{\sigma^2\psi(\sigma)}. \quad (103)$$

Using relation (63) and the fact that

$$\lim_{v\to\infty}\mu'(v)=0$$

and taking into account the second relation in (55) and inequalities (60), we obtain

$$\int_{b/\sigma}^\infty|\mu''(v)|dv=-\int_{b/\sigma}^\infty d\mu'(v)=\overline{\mu}'\left(\frac{b}{\sigma}\right)\frac{\psi(b)}{\psi(\sigma)}+\left|\overline{\mu}\left(\frac{b}{\sigma}\right)\right|\frac{\sigma|\psi'(b)|}{\psi(\sigma)}\le\frac{K_4}{\sigma^2\psi(\sigma)}. \quad (104)$$

It follows from (101)–(104) that

$$\left| \int_0^{\infty} \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| \le \frac{K}{t^2\sigma^2\psi(\sigma)}.$$

Then

$$I_2 = \frac{1}{\pi} \int_{|t|\ge\sigma} \left| \int_0^{\infty} \mu(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv \right| dt = O\left(\frac{1}{\sigma^3\psi(\sigma)}\right), \quad \sigma \to \infty. \tag{105}$$

Combining relations (91), (95), and (105), we conclude that the integral

$$A(\mu) = \int_{-\infty}^{\infty} |\hat{\mu}_\beta(t)|\, dt$$

satisfies the following estimate:

$$A(\mu) = O\left(\frac{1}{\sigma^3\psi(\sigma)}\right), \quad \sigma \to \infty. \tag{106}$$

Thus, the transform $\hat{\mu}_\beta(t)$ defined by (13) is summable on the real axis.

Since the transforms $\hat{\varphi}_\beta(t)$ and $\hat{\mu}_\beta(t)$ are summable on the entire number axis, the following relation is true:

$$\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}; B_\sigma\right)_{\hat{C}} = \sup_{f\in\hat{C}^{\psi}_{\beta,\infty}} \left\| \psi(\sigma) \int_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\sigma}\right) \hat{\varphi}_\beta(t) dt \right\|_{\hat{C}} + O\left(\psi(\sigma) A(\mu)\right).$$

Using relations (77) and (106), we obtain the asymptotic equality (80) as $\sigma \to \infty$.

Theorem 2 is proved.

Note that the conditions of Theorem 2 are satisfied, e.g., by the functions $\psi \in \mathfrak{A}$ that, for $v \ge 1$, have the following forms:

$$\psi(v) = \frac{\ln^{\alpha}(v+K)}{v^r} \quad \text{and} \quad \psi(v) = \frac{1}{v^r}(K + e^{-v}),$$

where

$$r > 4, \quad K > 0, \quad \text{and} \quad \alpha \in R,$$

and

$$\psi(v) = v^r e^{-Kv^{\alpha}},$$

where

$$\alpha > 0, \quad K > 0, \quad \text{and} \quad r \in R.$$

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