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APPROXIMATION OF (ψ, β)-DIFFERENTIABLE FUNCTIONS BY WEIERSTRASS INTEGRALS

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We obtain asymptotic equalities for upper bounds of approximations of functions from the classes $C^{\psi}_{\beta,\infty}$ and $L^{\psi}_{\beta,1}$ by Weierstrass integrals.

1. Main Definitions and Auxiliary Statements

Let C be the space of 2π-periodic continuous functions with norm $\|f\|_C = \max\limits_t \left|f(t)\right|$, let L_∞ be the space of 2π-periodic, measurable, essentially bounded functions with norm $\|f\|_\infty = \operatorname{ess\,sup}\limits_t \left|f(t)\right|$, and let L be the space of 2π-periodic functions summable on a period with norm

$$\|f\|_L = \|f\|_1 = \int\limits_{-\pi}^{\pi} \left|f(t)\right| dt.$$

In [1], classes of periodic functions were introduced as follows:

Let $f(x) \in L$ and let

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

be the Fourier series of f.

Further, let $\psi(k)$ be an arbitrary fixed function of natural argument and let β be a fixed real number. If the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k \cos\left(kx + \frac{\pi\beta}{2}\right) + b_k \sin\left(kx + \frac{\pi\beta}{2}\right)\right)$$

is the Fourier series of a certain summable function φ, then this function is called the (ψ, β)-derivative of $f(x)$ and is denoted by $f^{\psi}_{\beta}(x)$. The set of all functions $f(x)$ that satisfy this condition is denoted by L^{ψ}_{β}. The subset of continuous functions from L^{ψ}_{β} is denoted by C^{ψ}_{β}. If $f(x) \in L^{\psi}_{\beta}$ and $\left\|f^{\psi}_{\beta}(x)\right\|_1 \le 1$, then one says that $f(x)$ belongs to the class $L^{\psi}_{\beta,1}$; if $f(x) \in C^{\psi}_{\beta}$ and $\left\|f^{\psi}_{\beta}(x)\right\|_\infty \le 1$, then $f(x)$ belongs to the class $C^{\psi}_{\beta,\infty}$.

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For $\psi(k) = k^{-r}$, $r > 0$, the classes $C^{\psi}_{\beta,\infty}$ coincide with the classes W^{r}_{β} introduced by Nagy in [2], and $f^{\psi}_{\beta}(x) = f^{(r)}_{\beta}(x)$ is the (r,β)-derivative in the Weyl–Nagy sense. Furthermore, if $\beta = r$, $r \in N$, then f^{ψ}_{β} is the rth-order derivative of the function f, and the classes $C^{\psi}_{\beta,\infty}$ are the well-known Sobolev classes W^{r}.

Following Stepanets [1], we denote the set of convex-downward sequences $\psi(k)$ for which $\lim\limits_{k\to\infty} \psi(k) = 0$ by $\mathfrak{M}$. Without loss of generality, we assume that sequences $\psi(k)$ from the set $\mathfrak{M}$ are the restrictions of certain positive, continuous, convex-downward functions $\psi(t)$ of continuous argument $t \geq 1$ that tend to zero at infinity to the set of natural numbers. The set of these functions is also denoted by $\mathfrak{M}$. Thus,

$$\mathfrak{M} = \left\{ \psi(t)\colon \psi(t) > 0,\ \ \psi(t_1) - 2\psi\left(\frac{t_1+t_2}{2}\right) + \psi(t_2) \geq 0\ \ \forall t_1, t_2 \in [1,\infty),\ \ \lim_{t\to\infty} \psi(t) = 0 \right\}.$$

Let $\mathfrak{M}'$ denote the subset of functions $\psi(\cdot)$ from $\mathfrak{M}$ that satisfy the condition

$$\int\limits_{1}^{\infty} \frac{\psi(t)}{t} dt < \infty. \tag{2}$$

Further, we introduce the subset $\mathfrak{M}_0$ of the set $\mathfrak{M}$ by using the following characteristic: Let $\psi \in \mathfrak{M}$ and let $\eta(t) = \eta(\psi; t)$ be the function related to ψ by the equality

$$\eta(t) = \eta\left(\psi; t\right) = \psi^{-1}\left(\frac{1}{2}\psi(t)\right),$$

where ψ^{-1} is the function inverse to ψ. We set

$$\mu(t) = \mu(\psi; t) = \frac{t}{\eta(t) - t}.$$

Then

$$\mathfrak{M}_0 = \left\{ \psi \in \mathfrak{M}\colon 0 < \mu\left(\psi; t\right) \leq K\ \ \forall t \geq 1 \right\},$$

where K is a constant that may depend on ψ.

Let $f\left(x\right) \in L$. The quantity

$$W_{\delta}(f, x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-\frac{k^2}{\delta}} \left(a_k \cos kx + b_k \sin kx\right), \quad \delta > 0, \tag{3}$$

where a_k and b_k are the Fourier coefficients of the function f, is called the Weierstrass integral (see, e.g., [3, p. 150]).

The present paper is devoted to the investigation of the asymptotic behavior of the quantities

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; W_{\delta}\right)_C = \sup_{f \in C^{\psi}_{\beta,\infty}} \left\| f(x) - W_{\delta}(f, x) \right\|_C \tag{4}$$

and

$$\mathcal{E}\left(L^{\psi}_{\beta,1}; W_{\delta}\right)_1 = \sup_{f\in L^{\psi}_{\beta,1}} \left\|f(x) - W_{\delta}(f,x)\right\|_1 \tag{5}$$

as $\delta\to\infty$.

If a function $\varphi(\delta)=\varphi(\mathfrak{N};\delta)$ such that $\mathcal{E}(\mathfrak{N};W_\delta)_X=\varphi(\delta)+o(\varphi(\delta))$ as $\delta\to\infty$ is found in explicit form, then, following Stepanets [1, p. 198], we say that the Kolmogorov–Nikol'skii problem is solved for the class $\mathfrak{N}$ and the Weierstrass integral in the metric of the space X.

Note that the Kolmogorov–Nikol'skii problem for Weierstrass integrals on the classes W^r_β, W^r, etc., was studied by Bausov [4, 5], Bugrov [6], Baskakov [7], and Falaleev [8].

We set

$$\tau(u)=\tau_\delta(u,\psi)=\begin{cases}\left(1-e^{-u^2}\right)\dfrac{\psi(1)}{\psi(\sqrt{\delta})}, & 0\le u\le \dfrac{1}{\sqrt{\delta}},\\[2ex] \left(1-e^{-u^2}\right)\dfrac{\psi\left(\sqrt{\delta}u\right)}{\psi\left(\sqrt{\delta}\right)}, & u\ge\dfrac{1}{\sqrt{\delta}},\end{cases} \tag{6}$$

where $\psi(u)$ is a function defined and continuous for $u\ge 1$. Without loss of generality, we assume that the function $\psi(u)$ has the continuous second derivative on $[1;\infty)$.

In the present paper, we denote, generally speaking, different constants by K and K_i.

Below, we present several definitions and auxiliary statements due to Bausov [5] and Stepanets [1], which are used in what follows.

Definition 1 [5]. *Suppose that a function $\tau(u)$ is defined on $[0,\infty)$, absolutely continuous, and such that $\tau(\infty)=0$. One says that the function $\tau(u)$ belongs to $\mathcal{E}_a$ if the derivative $\tau'(u)$ can be extended to the points where it does not exist so that the integrals $\int_0^{\frac{a}{2}} u|d\tau'(u)|$ and $\int_{\frac{a}{2}}^{\infty}|u-a||d\tau'(u)|$ exist for a certain $a\ge 0$.*

Theorem 1′ [5, p. 24]. *Suppose that $\tau(u)\in\mathcal{E}_a$ and $\sin\frac{\beta\pi}{2}\tau(0)=0$. Then, for the convergence of the integral*

$$A(\tau)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\int_0^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt, \tag{7}$$

it is necessary and sufficient that the integrals

$$\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\infty}\frac{|\tau(u)|}{u}du \qquad \text{and} \qquad \int_0^{a}\frac{|\tau(a-u)-\tau(a+u)|}{u}du$$

be convergent. Moreover, the following estimate is true:

$$\left|A(\tau)-\frac{4}{\pi^2}\int_0^{\infty}\xi\left(\sin\frac{\beta\pi}{2}\tau(u), j_u\left[\tau(a-u)-\tau(a+u)\right]\right)\frac{du}{u}\right|\le KH(\tau),$$

where $\xi(A,B)$ *is the function introduced in [9] as follows:*

$$\xi(A,B)=\begin{cases}\dfrac{\pi}{2}|A|, & |B|\le|A|,\\[2mm] |A|\arcsin\left|\dfrac{A}{B}\right|+\sqrt{B^2-A^2}, & |B|>|A|,\end{cases}\qquad j_u=\begin{cases}1, & 0<u<a,\\[2mm] 0, & u\ge a,\end{cases}$$

$$H(\tau)=|\tau(0)|+|\tau(a)|+\int\limits_0^{\frac{a}{2}}u\left|d\tau'(u)\right|+\int\limits_{\frac{a}{2}}^{\infty}|u-a|\left|d\tau'(u)\right|. \tag{8}$$

If

$$\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\int\limits_0^{\infty}\frac{|\tau(u)|}{u}du\ge\frac{4}{\pi^2}\int\limits_0^{a}\frac{|\tau(a-u)-\tau(a+u)|}{u}du,$$

then

$$\left|A(\tau)-\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\int\limits_0^{\infty}\frac{|\tau(u)|}{u}du\right|\le K\left(\int\limits_0^{a}\frac{|\tau(a-u)-\tau(a+u)|}{u}du+H(\tau)\right); \tag{9}$$

if

$$\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\int\limits_0^{\infty}\frac{|\tau(u)|}{u}du\le\frac{4}{\pi^2}\int\limits_0^{a}\frac{|\tau(a-u)-\tau(a+u)|}{u}du,$$

then

$$\left|A(\tau)-\frac{4}{\pi^2}\int\limits_0^{a}\frac{|\tau(a-u)-\tau(a+u)|}{u}du\right|\le K\left(\left|\sin\frac{\beta\pi}{2}\right|\int\limits_0^{\infty}\frac{|\tau(u)|}{u}du+H(\tau)\right). \tag{10}$$

Theorem 2′ [1, p. 161]**.** *A function* $\psi\in\mathfrak{M}$ *belongs to* $\mathfrak{M}_0$ *if and only if the quantity*

$$\alpha(t)=\frac{\psi(t)}{t\,|\psi'(t)|},\qquad \psi'(t):=\psi'(t+0), \tag{11}$$

satisfies the condition $\alpha(t)\ge K>0\ \ \forall t\ge 1.$

Theorem 3′ [1, p. 175]**.** *For a function* $\psi\in\mathfrak{M}$ *to belong to* $\mathfrak{M}_0,$ *it is necessary and sufficient that there exist a constant* K *such that the following inequality holds for all* $t\ge 1:$

$$\frac{\psi(t)}{\psi(ct)}\le K,$$

where c *is an arbitrary constant that satisfies the condition* $c>1.$

2. Asymptotic Estimates for Upper Bounds of Deviations of Weierstrass Integrals from Functions of the Classes $C^{\psi}_{\beta,\infty}$

We need the following analog of Lemma 1 in [10]:

Lemma 1. *If, for the function $\tau(u)$ defined by (6), its transform*

$$\hat{\tau}_\beta(t) = \hat{\tau}(t,\beta) = \frac{1}{\pi}\int\limits_0^\infty \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du \tag{12}$$

is summable on the entire number axis, then the following equality is true:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; W_\delta\right)_C = \psi(\sqrt{\delta})A(\tau) + O\left(\psi\left(\sqrt{\delta}\right)\int\limits_{|t|\ge\frac{\sqrt{\delta}\,\pi}{2}} |\hat{\tau}_\beta(t)|\,dt\right), \tag{13}$$

where $A(\tau)$ is defined by (7).

Theorem 1. *Suppose that $\psi \in \mathfrak{M}'_0 = \mathfrak{M}_0 \cap \mathfrak{M}'$ and the function $g(u) = u^2\psi(u)$ is convex upward or downward on $[b;\infty)$, $b \ge 1$. Then the following equality holds as $\delta \to \infty$:*

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; W_\delta\right)_C = \psi(\sqrt{\delta})A(\tau) + O\left(\frac{1}{\delta} + \frac{\psi(\sqrt{\delta})}{\sqrt{\delta}}\right), \tag{14}$$

where $A(\tau)$ is defined by (7) and satisfies the following estimate:

$$A(\tau) = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\left(\frac{1}{\delta\psi(\sqrt{\delta})}\int\limits_1^{\sqrt{\delta}} u\psi(u)du + \frac{1}{\psi(\sqrt{\delta})}\int\limits_{\sqrt{\delta}}^\infty \frac{\psi(u)}{u}du\right) + O\left(1 + \frac{1}{\delta\psi(\sqrt{\delta})}\right). \tag{15}$$

Proof. We verify whether the conditions of Lemma 1 are satisfied. To this end, we establish the summability of the transform (12) of the function $\tau(u)$, i.e., the convergence of integral (7). Using Theorem $1'$, we estimate the following integrals:

$$\int\limits_0^{\frac{1}{2}} u\left|d\tau'(u)\right|, \qquad \int\limits_{\frac{1}{2}}^\infty |u-1|\left|d\tau'(u)\right|, \tag{16}$$

$$\left|\sin\frac{\beta\pi}{2}\right|\int\limits_0^\infty \frac{|\tau(u)|}{u}du, \qquad \int\limits_0^1 \frac{|\tau(1-u)-\tau(1+u)|}{u}du. \tag{17}$$

To estimate the first integral in (16), we divide the segment $\left[0;\frac{1}{2}\right]$ into the two parts $\left[0;\frac{1}{\sqrt{\delta}}\right]$ and $\left[\frac{1}{\sqrt{\delta}};\frac{1}{2}\right]$ (for $\delta > 4b^2$).

Taking into account that $\tau''(u) \geq 0$ on $\left[0;\frac{1}{\sqrt{\delta}}\right]$ and using the inequality

$$e^{-u^2} \leq 1, \quad u \in R, \tag{18}$$

we get

$$\int\limits_0^{\frac{1}{\sqrt{\delta}}} u\left|d\tau'(u)\right| = \frac{\psi(1)}{\psi(\sqrt{\delta})}\left(\frac{2}{\delta}e^{-\frac{1}{\delta}} - 1 + e^{-\frac{1}{\delta}}\right) = O\left(\frac{1}{\delta\psi(\sqrt{\delta})}\right), \quad \delta\to\infty. \tag{19}$$

Now let $u \in \left[\frac{1}{\sqrt{\delta}};\frac{1}{2}\right]$. Setting

$$\tau_1(u) = \left(1 - e^{-u^2} - u^2\right)\frac{\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}, \qquad \tau_2(u) = u^2\frac{\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}, \tag{20}$$

we obtain

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u\left|d\tau'(u)\right| \leq \int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u\left|d\tau_1'(u)\right| + \int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u\left|d\tau_2'(u)\right|. \tag{21}$$

Let us estimate the first integral on the right-hand side of (21). Since

$$\tau_1''(u) = \left(1 - e^{-u^2} - u^2\right)\frac{\delta\psi''(\sqrt{\delta}u)}{\psi(\sqrt{\delta})} + 4u\left(e^{-u^2} - 1\right)\frac{\sqrt{\delta}\psi'(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}$$

$$+ 2\left(e^{-u^2} - 2u^2e^{-u^2} - 1\right)\frac{\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}, \tag{22}$$

taking into account the inequalities

$$e^{-u^2} + u^2 - 1 \leq \frac{u^4}{2}, \qquad 1 - e^{-u^2} \leq u^2,$$

$$2u^2e^{-u^2} - e^{-u^2} + 1 \leq 3u^2, \quad u \in R, \tag{23}$$

we get

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u|d\tau_1'(u)| \leq \frac{\delta}{2\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u^5\psi''(\sqrt{\delta}u)du + \frac{4\sqrt{\delta}}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u^4|\psi'(\sqrt{\delta}u)|du + \frac{6}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u^3\psi(\sqrt{\delta}u)du.$$

Integrating the first integral on the right–hand side of the last inequality by parts and using Theorems $2'$ and $3'$, we obtain

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u|d\tau_1'(u)| \leq \frac{\sqrt{\delta}\left|\psi'\left(\frac{\sqrt{\delta}}{2}\right)\right|}{2^6\psi(\sqrt{\delta})} + \frac{|\psi'(1)|}{2\delta^2\psi(\sqrt{\delta})} + \frac{13\sqrt{\delta}}{2\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u^4|\psi'(\sqrt{\delta}u)|du + \frac{6}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u^3\psi(\sqrt{\delta}u)du$$

$$\leq K_1 + \frac{K_2}{\delta^2\psi(\sqrt{\delta})} + \frac{K_3}{\psi(\sqrt{\delta})}\left(\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} + \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{1}{2}}\right) u^3\psi(\sqrt{\delta}u)du. \tag{24}$$

Here and in what follows, we assume that $\psi'(1) = \psi'(1+0)$.

Since the function $g(u) = u^2\psi(u)$ is bounded on $[1;b]$, we have

$$\frac{1}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} u^3\psi(\sqrt{\delta}u)du = \frac{1}{\delta^2\psi(\sqrt{\delta})}\int\limits_1^b u^3\psi(u)\,du$$

$$\leq \frac{K}{\delta^2\psi(\sqrt{\delta})}\int\limits_1^b u\,du = O\left(\frac{1}{\delta^2\psi(\sqrt{\delta})}\right), \quad \delta\to\infty. \tag{25}$$

Taking into account that the function $g(u) = u^2\psi(u)$ is convex upward or downward for $u \geq b$, we get

$$\int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{1}{2}} u^3\psi(\sqrt{\delta}u)du \leq \int\limits_{\frac{b}{\sqrt{\delta}}}^{1} u^3\psi(\sqrt{\delta}u)du$$

$$\leq \frac{1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}\int\limits_b^{\sqrt{\delta}} u^2\psi(u)du = O\left(1 + \frac{1}{\delta\psi(\sqrt{\delta})}\right), \quad \delta\to\infty. \tag{26}$$

Using (25) and (26), we obtain the following relation from (24):

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{1}{2}} u|d\tau_1'(u)| = O\left(1 + \frac{1}{\delta\psi(\sqrt{\delta})}\right), \quad \delta\to\infty. \tag{27}$$

Let us estimate the second integral on the right-hand side of (21). Taking into account that, for $u \geq \frac{1}{\sqrt{\delta}}$, one has

$$\tau_2''(u) = 2\frac{\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})} + 4u\frac{\sqrt{\delta}\psi'\left(\sqrt{\delta}u\right)}{\psi(\sqrt{\delta})} + u^2\frac{\delta\psi''(\sqrt{\delta}u)}{\psi(\sqrt{\delta})},$$

we obtain the following relation for $\delta > 4b^2$:

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} u\left|d\tau_2'(u)\right| \leq \frac{\delta}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} u^3\psi''(\sqrt{\delta}u)\,du + \frac{4\sqrt{\delta}}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} u^2\left|\psi'(\sqrt{\delta}u)\right|\,du + \frac{2}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} u\psi(\sqrt{\delta}u)\,du.$$

Integrating the first and the second integral on the right-hand side of the last inequality twice and once, respectively, by parts and taking into account that the function $\psi(u)$ decreases on $[1;\infty)$, we obtain

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} u\left|d\tau_2'(u)\right| \leq \frac{\sqrt{\delta}}{\psi(\sqrt{\delta})}u^3\psi'\left(\sqrt{\delta}u\right)\Bigg|_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} - \frac{7}{\psi(\sqrt{\delta})}u^2\psi(\sqrt{\delta}u)\Bigg|_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} + \frac{16}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} u\psi(\sqrt{\delta}u)du$$

$$= O\left(\frac{1}{\delta\psi(\sqrt{\delta})}\right). \tag{28}$$

Since the function $g(u) = u^2\psi(u)$ is convex on $[b;\infty)$, we have

$$\int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{1}{2}} u|d\tau_2'(u)| = \left|\int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{1}{2}} ud\tau_2'(u)\right| = \left|\left(u\tau_2'(u) - \tau_2(u)\right)\Bigg|_{\frac{b}{\sqrt{\delta}}}^{\frac{1}{2}}\right| = O\left(1 + \frac{1}{\delta\psi(\sqrt{\delta})}\right). \tag{29}$$

Thus, it follows from (19), (21), and (27)–(29) that

$$\int\limits_0^{\frac{1}{2}} u\left|d\tau'(u)\right| = O\left(1 + \frac{1}{\delta\psi(\sqrt{\delta})}\right), \quad \delta\to\infty. \tag{30}$$

We estimate the second integral in (16). Taking into account that, according to (6), for $u \geq \frac{1}{\sqrt{\delta}}$ one has

$$\tau''(u) = \left(1 - e^{-u^2}\right)\frac{\delta\psi''(\sqrt{\delta}u)}{\psi(\sqrt{\delta})} + 4ue^{-u^2}\frac{\sqrt{\delta}\psi'(\sqrt{\delta}u)}{\psi(\sqrt{\delta})} + \; 2\left(e^{-u^2} - 2u^2e^{-u^2}\right)\frac{\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})} \tag{31}$$

and using the fact that $|u-1| \leq u$, $u \in \left[\frac{1}{2};\infty\right)$, and

$$1 - e^{-u^2} \leq 1, \qquad u^2e^{-u^2} \leq 1, \qquad \left|u - 2u^3\right|e^{-u^2} \leq \frac{2}{u^2}, \quad u \in R, \tag{32}$$

we get

$$\int\limits_{\frac{1}{2}}^{\infty} |u-1||d\tau'(u)| \leq \frac{\delta}{\psi(\sqrt{\delta})} \int\limits_{\frac{1}{2}}^{\infty} u\psi''\big(\sqrt{\delta}u\big)du + \frac{4\sqrt{\delta}}{\psi(\sqrt{\delta})} \int\limits_{\frac{1}{2}}^{\infty} \big|\psi'\big(\sqrt{\delta}u\big)\big|du + \frac{4}{\psi\big(\sqrt{\delta}\big)} \int\limits_{\frac{1}{2}}^{\infty} \frac{\psi(\sqrt{\delta}u)}{u^2}du. \tag{33}$$

Integrating the first integral on the right-hand side of (33) by parts and using Theorems $2'$ and $3'$, we obtain

$$\int\limits_{\frac{1}{2}}^{\infty} |u-1| \left|d\tau'(u)\right| = O(1), \quad \delta \to \infty. \tag{34}$$

To estimate the first integral in (17), we divide the interval $[0, \infty)$ into the three parts $\left[0, \frac{1}{\sqrt{\delta}}\right]$, $\left[\frac{1}{\sqrt{\delta}}, 1\right]$, and $[1, \infty)$.

Let $u \in \left[0, \frac{1}{\sqrt{\delta}}\right]$. Taking into account (6) and the second relation in (23), we get

$$\int\limits_{0}^{\frac{1}{\sqrt{\delta}}} \frac{|\tau(u)|}{u}du = \frac{\psi(1)}{\psi(\sqrt{\delta})} \int\limits_{0}^{\frac{1}{\sqrt{\delta}}} (1-e^{-u^2})\frac{du}{u} \leq \frac{\psi(1)}{\psi(\sqrt{\delta})} \int\limits_{0}^{\frac{1}{\sqrt{\delta}}} udu \leq \frac{K}{\delta\psi(\sqrt{\delta})}. \tag{35}$$

According to (6), for $u \in \left[\frac{1}{\sqrt{\delta}}, 1\right]$ we have

$$\left|\int\limits_{\frac{1}{\sqrt{\delta}}}^{1} \frac{\tau(u)}{u}du - \frac{1}{\psi(\sqrt{\delta})} \int\limits_{\frac{1}{\sqrt{\delta}}}^{1} u\psi(\sqrt{\delta}u)du\right| \leq \frac{1}{\psi(\sqrt{\delta})} \int\limits_{\frac{1}{\sqrt{\delta}}}^{1} \frac{|1-e^{-u^2}-u^2|}{u}\psi(\sqrt{\delta}u)du.$$

By virtue of the first inequality in (23) and estimates (25) and (26), we get

$$\left|\int\limits_{\frac{1}{\sqrt{\delta}}}^{1} \frac{\tau(u)}{u}du - \frac{1}{\psi(\sqrt{\delta})} \int\limits_{\frac{1}{\sqrt{\delta}}}^{1} u\psi(\sqrt{\delta}u)du\right| \leq \frac{1}{2\psi(\sqrt{\delta})} \int\limits_{\frac{1}{\sqrt{\delta}}}^{1} u^3\psi(\sqrt{\delta}u)du$$

$$= \frac{1}{2\psi(\sqrt{\delta})} \int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} u^3\psi(\sqrt{\delta}u)du + \frac{1}{2\psi(\sqrt{\delta})} \int\limits_{\frac{b}{\sqrt{\delta}}}^{1} u^3\psi(\sqrt{\delta}u)du$$

$$= O\left(1 + \frac{1}{\delta\psi(\sqrt{\delta})}\right).$$

It follows from the last relations that

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{1} \frac{|\tau(u)|}{u}du = \frac{1}{\delta\psi(\sqrt{\delta})}\int\limits_{1}^{\sqrt{\delta}} u\psi(u)du + O\left(1+\frac{1}{\delta\psi(\sqrt{\delta})}\right), \quad \delta\to\infty. \tag{36}$$

Finally, let $u\in[1,\infty)$. Since

$$\left|\int\limits_{1}^{\infty}\frac{\tau(u)}{u}du - \frac{1}{\psi(\sqrt{\delta})}\int\limits_{1}^{\infty}\frac{\psi(\sqrt{\delta}u)}{u}du\right| \le \frac{1}{\psi(\sqrt{\delta})}\int\limits_{1}^{\infty}\frac{e^{-u^2}}{u}\psi(\sqrt{\delta}u)du \le K,$$

we have

$$\int\limits_{1}^{\infty}\frac{|\tau(u)|}{u}du = \frac{1}{\psi(\sqrt{\delta})}\int\limits_{\sqrt{\delta}}^{\infty}\frac{\psi(u)}{u}du + O(1). \tag{37}$$

Combining relations (35)–(37), we get

$$\int\limits_{0}^{\infty}\frac{|\tau(u)|}{u}du = \frac{1}{\delta\psi(\sqrt{\delta})}\int\limits_{1}^{\sqrt{\delta}} u\psi(u)du + \frac{1}{\psi(\sqrt{\delta})}\int\limits_{\sqrt{\delta}}^{\infty}\frac{\psi(u)}{u}du + O\left(1+\frac{1}{\delta\psi(\sqrt{\delta})}\right). \tag{38}$$

Let us estimate the second integral in (17). For the function $\tau(u)$ defined by (6), according to Lemma 1 in [11] the following equality holds for all $\psi\in\mathfrak{M}_0$:

$$\int\limits_{0}^{1}\frac{|\tau(1-u)-\tau(1+u)|}{u}du = \int\limits_{0}^{1}\frac{|\lambda(1-u)-\lambda(1+u)|}{u}du + O\big(H(\tau)\big), \tag{39}$$

where $H(\tau)$ is defined by (8) and $\lambda(u)=e^{-u^2}$.

Using the fact that

$$\int\limits_{0}^{1}\frac{|\lambda(1-u)-\lambda(1+u)|}{u}du = \int\limits_{0}^{1}\frac{e^{-(1-u)^2}-e^{-(1+u)^2}}{u}du = O(1)$$

and relations (30), (34), and (39), we obtain

$$\int\limits_{0}^{1}\frac{|\tau(1-u)-\tau(1+u)|}{u}du = O\left(1+\frac{1}{\delta\psi(\sqrt{\delta})}\right), \quad \delta\to\infty. \tag{40}$$

Thus, taking into account relations (30), (34), (38), and (40) and using Theorem 1′, we establish that the transform (12) of the function $\tau(u)$ is summable on the entire number axis. Therefore, by virtue of Lemma 1, equality (13) is true. Using inequalities (9) and (10) and taking into account (30), (34), (38), and (40), we obtain relation (15).

Let us estimate the remainder on the right-hand side of (13). We have

$$\hat{\tau}_\beta(t) = \frac{1}{\pi}\int\limits_0^\infty \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du = \frac{1}{\pi}\left(\int\limits_0^{\frac{1}{\sqrt{\delta}}} + \int\limits_{\frac{1}{\sqrt{\delta}}}^\infty\right)\tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du. \tag{41}$$

Integrating the integrals on the right-hand side of equality (41) twice by parts, we get

$$\int\limits_0^{\frac{1}{\sqrt{\delta}}} \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du = \frac{1}{t}\left(1 - e^{-\frac{1}{\delta}}\right)\frac{\psi(1)}{\psi(\sqrt{\delta})}\sin\left(\frac{1}{\sqrt{\delta}}t + \frac{\beta\pi}{2}\right)$$

$$+ \frac{1}{t^2}\frac{2e^{-\frac{1}{\delta}}\psi(1)}{\sqrt{\delta}\psi(\sqrt{\delta})}\cos\left(\frac{1}{\sqrt{\delta}}t + \frac{\beta\pi}{2}\right) - \frac{1}{t^2}\int\limits_0^{\frac{1}{\sqrt{\delta}}} \tau''(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du, \tag{42}$$

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^\infty \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du = -\frac{1}{t}\left(1 - e^{-\frac{1}{\delta}}\right)\frac{\psi(1)}{\psi(\sqrt{\delta})}\sin\left(\frac{1}{\sqrt{\delta}}t + \frac{\beta\pi}{2}\right)$$

$$-\frac{1}{t^2}\left(\frac{2e^{-\frac{1}{\delta}}\psi(1)}{\sqrt{\delta}\psi(\sqrt{\delta})} + \left(1 - e^{-\frac{1}{\delta}}\right)\frac{\sqrt{\delta}\psi'(1)}{\psi(\sqrt{\delta})}\right)\cos\left(\frac{1}{\sqrt{\delta}}t + \frac{\beta\pi}{2}\right)$$

$$-\frac{1}{t^2}\int\limits_{\frac{1}{\sqrt{\delta}}}^\infty \tau''(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du. \tag{43}$$

Substituting (42) and (43) into (41), we obtain

$$\frac{1}{\pi}\int\limits_0^\infty \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du = -\frac{1}{\pi t^2}\int\limits_0^{\frac{1}{\sqrt{\delta}}} \tau''(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du - \frac{1}{\pi t^2}\int\limits_{\frac{1}{\sqrt{\delta}}}^\infty \tau''(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du$$

$$-\frac{1}{\pi t^2}\left(1 - e^{-\frac{1}{\delta}}\right)\frac{\sqrt{\delta}\psi'(1)}{\psi(\sqrt{\delta})}\cos\left(\frac{1}{\sqrt{\delta}}t + \frac{\beta\pi}{2}\right),$$

whence

$$\left|\frac{1}{\pi}\int_0^\infty \tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right| \le \frac{1}{\pi t^2}\left(\int_0^{\frac{1}{\sqrt{\delta}}}+\int_{\frac{1}{\sqrt{\delta}}}^1+\int_1^\infty\right)|\tau''(u)|du+\frac{1}{t^2}\frac{K}{\sqrt{\delta}\psi(\sqrt{\delta})}. \qquad (44)$$

Taking into account that $\tau''(u)\ge 0$ on $\left[0,\frac{1}{\sqrt{\delta}}\right]$ and using inequality (18), we get

$$\int_0^{\frac{1}{\sqrt{\delta}}}|\tau''(u)|du=\int_0^{\frac{1}{\sqrt{\delta}}}\tau''(u)du=\frac{2\psi(1)}{\sqrt{\delta}\psi(\sqrt{\delta})}e^{-\frac{1}{\delta}}=O\left(\frac{1}{\sqrt{\delta}\psi(\sqrt{\delta})}\right),\quad \delta\to\infty. \qquad (45)$$

Let $u\in\left[\frac{1}{\sqrt{\delta}},1\right]$. Reasoning as in the estimation of the first integral in (16) on the segment $\left[\frac{1}{\sqrt{\delta}},\frac{1}{2}\right]$ [see (20)–(29)], we obtain the following estimate:

$$\int_{\frac{1}{\sqrt{\delta}}}^1|\tau''(u)|du=O\left(1+\frac{1}{\sqrt{\delta}\psi(\sqrt{\delta})}\right),\quad \delta\to\infty. \qquad (46)$$

Now let $u\in[1,\infty)$. Using equality (31), the first inequality in (32), the inequalities

$$ue^{-u^2}\le 1,\qquad (2u^2-1)e^{-u^2}\le\frac{1}{u^2},\quad u\in R,$$

and Theorem 3′, we get

$$\int_1^\infty|\tau''(u)|du\le\frac{\delta}{\psi(\sqrt{\delta})}\int_1^\infty\psi''(\sqrt{\delta}u)du+\frac{4\sqrt{\delta}}{\psi(\sqrt{\delta})}\int_1^\infty\left|\psi'(\sqrt{\delta}u)\right|du+\frac{1}{\psi(\sqrt{\delta})}\int_1^\infty\frac{\psi(\sqrt{\delta}u)}{u^2}du=O(1). \qquad (47)$$

Combining relations (45)–(47), we deduce the following result from (44):

$$\left|\frac{1}{\pi}\int_0^\infty\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|=O\left(1+\frac{1}{\sqrt{\delta}\psi(\sqrt{\delta})}\right)\frac{1}{t^2}.$$

Hence,

$$\int_{|t|\ge\frac{\sqrt{\delta}\pi}{2}}\left|\hat{\tau}_\beta(t)\right|dt=O\left(\frac{1}{\delta\psi(\sqrt{\delta})}+\frac{1}{\sqrt{\delta}}\right). \qquad (48)$$

It follows from (48) and (13) that equality (14) is true.

Theorem 1 is proved.

It should be noted that, for $\psi(u)=\dfrac{1}{u^r},\ r<2,$ Theorem 1 was proved by Bausov in [5, p. 31].

Corollary 1. *If the conditions of Theorem 1 are satisfied,* $\sin\dfrac{\beta\pi}{2}\neq 0,$ *and* $\lim\limits_{t\to\infty}\alpha(t)=\infty,$ *where* $\alpha(t)$ *is defined by (11), then, as* $\delta\to\infty,$ *the following asymptotic equality is true:*

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty};W_\delta\right)_C=\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\int\limits_{\sqrt{\delta}}^{\infty}\frac{\psi(u)}{u}du+O\left(\psi(\sqrt{\delta})\right). \tag{49}$$

Proof. If $\psi\in\mathfrak{M}_0'$ and $\lim\limits_{t\to\infty}\alpha(t)=\infty,$ then, for any $\varepsilon>0,$ there exists $u_0\geq 1$ such that $(u^\varepsilon\psi(u))'>0$ for $u>u_0,$ i.e., the function $u^\varepsilon\psi(u)$ increases beginning with a certain number $u_0,$ and $\lim\limits_{u\to\infty}u^\varepsilon\psi(u)=\infty.$ Thus, for sufficiently large δ and $0<\varepsilon<2,$ we have

$$\frac{1}{\delta\psi(\sqrt{\delta})}\int\limits_1^{\sqrt{\delta}}u\psi(u)du\leq\frac{(\sqrt{\delta})^\varepsilon\psi(\sqrt{\delta})}{\delta\psi(\sqrt{\delta})}\int\limits_1^{\sqrt{\delta}}\frac{du}{u^{\varepsilon-1}}=O(1). \tag{50}$$

Using the l'Hospital rule and the fact that $\lim\limits_{t\to\infty}\alpha(t)=\infty,$ we get

$$\lim_{x\to\infty}\frac{\int\limits_x^{\infty}\frac{\psi(u)}{u}du}{\psi(x)}=\lim_{x\to\infty}\frac{\psi(x)}{x|\psi'(x)|}=\infty. \tag{51}$$

Taking into account that

$$\frac{1}{\delta}+\frac{\psi(\sqrt{\delta})}{\sqrt{\delta}}=o\left(\psi(\sqrt{\delta})\right) \tag{52}$$

and using relations (50) and (51), we deduce (49) from (14) and (15).

Functions of the form $\psi(u)=\dfrac{1}{\ln^\alpha(u+K)},$ where $\alpha>1$ and $K>0,$ can serve as an example of functions satisfying the conditions of Corollary 1.

Corollary 2. *Suppose that* $\psi\in\mathfrak{M}_0,\ \sin\dfrac{\beta\pi}{2}\neq 0,$ *the function* $u^2\psi(u)$ *is convex upward or downward for* $u\geq b\geq 1,$ *and*

$$\lim_{u\to\infty}u^2\psi(u)=\infty, \tag{53}$$

$$\lim_{\delta\to\infty}\frac{1}{\delta\psi(\sqrt{\delta})}\int\limits_1^{\sqrt{\delta}}u\psi(u)du=\infty. \tag{54}$$

Then the following asymptotic equality holds as $\delta \to \infty$:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; W_\delta\right)_C = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta}\int\limits_1^{\sqrt{\delta}} u\psi(u)du + O\left(\psi(\sqrt{\delta})\right). \tag{55}$$

Proof. If the function ψ satisfies conditions (53) and (54), then, using the l'Hospital rule, we get

$$\lim_{x\to\infty}\frac{\int\limits_1^x u\psi(u)}{x^2\psi(x)} = \lim_{x\to\infty}\frac{x\psi(x)}{2x\psi(x)+x^2\psi'(x)} = \frac{1}{2+\lim\limits_{x\to\infty}\dfrac{x\psi'(x)}{\psi(x)}} = \infty.$$

Hence,

$$\lim_{x\to\infty}\frac{x\psi'(x)}{\psi(x)} = -2. \tag{56}$$

Taking (51) and (56) into account, we obtain

$$\int\limits_{\sqrt{\delta}}^{\infty}\frac{\psi(u)}{u}du = O\left(\psi(\sqrt{\delta})\right).$$

Using the last estimate and relations (14), (15), and (52)–(54), we get (55).

Note that the conditions of Corollary 2 are satisfied, e.g., by functions of the form $\psi(u) = \dfrac{1}{u^2}\ln^\alpha(u+K)$, where $K > 0$ and $\alpha > 0$.

Corollary 3. *Suppose that* $\psi \in \mathfrak{M}_0$, $\sin\dfrac{\beta\pi}{2} \neq 0$, *the function* $u^2\psi(u)$ *is convex downward for* $u \geq b \geq 1$, *and*

$$\lim_{u\to\infty} u^2\psi(u) = K < \infty, \tag{57}$$

$$\lim_{\delta\to\infty}\int\limits_1^{\sqrt{\delta}} u\psi(u)du = \infty. \tag{58}$$

Then the following asymptotic equality holds as $\delta \to \infty$:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; W_\delta\right)_C = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta}\int\limits_1^{\sqrt{\delta}} u\psi(u)du + O\left(\frac{1}{\delta}\right). \tag{59}$$

Proof. Since the function $u^2\psi(u)$ is convex downward on the interval $[b;\infty)$, $b \geq 1$, and satisfies condition (57), we conclude that this function is monotonically decreasing for $u \geq b$. Thus, for $\delta > b^2$, we have

$$\int\limits_{\sqrt{\delta}}^{\infty} \frac{\psi(u)}{u} du \leq \int\limits_{\sqrt{\delta}}^{\infty} \frac{u^2\psi(u)}{u^3} du \leq \delta\psi(\sqrt{\delta}) \int\limits_{\sqrt{\delta}}^{\infty} \frac{1}{u^3} du = O\left(\psi(\sqrt{\delta})\right),$$

$$\psi(\sqrt{\delta}) = O\left(\frac{1}{\delta}\right).$$

Using the last estimates and relations (14), (15), (57), and (58), we get (59).

The functions

$$\psi(u) = \frac{1}{u^2}(K + e^{-u}) \qquad \text{and} \qquad \psi(u) = \frac{1}{u^2 \ln^{\alpha}(u+K)},$$

where $K > 0$ and $0 \leq \alpha \leq 1,$ can serve as examples of functions ψ satisfying the conditions of Corollary 3.

In particular, for $\psi(u) = \dfrac{1}{u^2},$ we derive the following asymptotic equality from (59):

$$\mathcal{E}\left(W^2_{\beta}; W_{\delta}\right)_C = \frac{1}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{\ln\delta}{\delta} + O\left(\frac{1}{\delta}\right);$$

this asymptotic equality was obtained by Bausov in [5, p. 31].

Note that, under the conditions of Corollaries 1–3, equalities (49), (55), and (59) give a solution of the Kolmogorov–Nikol'skii problem for the Weierstrass integrals W_{δ} on the classes $C^{\psi}_{\beta,\infty}$ in the uniform metric.

Let G be the set of functions $\psi \in \mathfrak{M}$ that satisfy the following condition: For any constant $K > 0,$ there exists a point $u_0 = u_0(K) \geq 1$ such that, for $u > u_0,$ a function $\alpha(u)$ of the form (11) satisfies the inequality

$$\alpha(u) < \frac{1}{2}\left(1 - \frac{K}{u^2}\right).$$

Theorem 2. *Suppose that* $\psi \in G,$ *the function* $g(u) = u^2\psi(u)$ *is convex downward on* $[b;\infty),$ $b \geq 1,$ *and*

$$\int\limits_{1}^{\infty} u\psi(u) du < \infty. \tag{60}$$

Then the following asymptotic equality holds as $\delta \to \infty:$

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; W_{\delta}\right)_C = \frac{1}{\delta} \sup_{f \in C^{\psi}_{\beta,\infty}} \left\| f^{(2)}_0(x) \right\|_C + O\left(\frac{1}{\delta\sqrt{\delta}} \int\limits_{1}^{\sqrt{\delta}} t^2\psi(t)dt + \frac{1}{\delta}\int\limits_{\sqrt{\delta}}^{\infty} t\psi(t)dt\right), \tag{61}$$

where $f^{(2)}_0(x)$ *is the Weyl–Nagy* (r,β)*-derivative for* $r = 2$ *and* $\beta = 0.$

Proof. We represent the function $\tau(u)$ defined by (6) in the form $\tau(u) = \varphi(u) + \mu(u),$ where

$$\varphi(u) = \begin{cases} u^2 \dfrac{\psi(1)}{\psi(\sqrt{\delta})}, & 0 \le u \le \dfrac{1}{\sqrt{\delta}}, \\ \\ u^2 \dfrac{\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}, & u \ge \dfrac{1}{\sqrt{\delta}}, \end{cases} \tag{62}$$

$$\mu(u) = \begin{cases} (1 - e^{-u^2} - u^2) \dfrac{\psi(1)}{\psi(\sqrt{\delta})}, & 0 \le u \le \dfrac{1}{\sqrt{\delta}}, \\ \\ (1 - e^{-u^2} - u^2) \dfrac{\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}, & u \ge \dfrac{1}{\sqrt{\delta}}. \end{cases} \tag{63}$$

Let us verify that the transforms $\hat{\varphi}_\beta(t)$ and $\hat{\mu}_\beta(t)$ of the functions $\varphi(u)$ and $\mu(u)$ are summable [see (12)]. We show that the integral

$$A(\varphi) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \left| \int\limits_0^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2} \right) du \right| dt$$

is convergent. Using condition (60) and the fact that the function $g(u)$ is convex downward, one can easily verify that

$$\lim_{u\to\infty} u^2 \psi(u) = 0 \qquad \text{and} \qquad \lim_{u\to\infty} u^2 \psi'(u) = 0.$$

Integrating twice by parts and taking into account that $\varphi(0) = \varphi'(0) = 0$ and

$$\lim_{u\to\infty} \varphi(u) = \lim_{u\to\infty} \varphi'(u) = 0,$$

we obtain

$$\int\limits_0^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2} \right) du = \left(\int\limits_0^{\frac{1}{\sqrt{\delta}}} + \int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty} \right) \varphi(u) \cos\left(ut + \frac{\beta\pi}{2} \right) du$$

$$= -\frac{1}{t^2} \left(\int\limits_0^{\frac{1}{\sqrt{\delta}}} + \int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty} \right) \varphi''(u) \cos\left(ut + \frac{\beta\pi}{2} \right) du - \frac{1}{t^2} \frac{\psi'(1)}{\sqrt{\delta}\psi(\sqrt{\delta})} \cos\left(\frac{1}{\sqrt{\delta}} t + \frac{\beta\pi}{2} \right),$$

whence

$$\left|\int_0^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|\le\frac{1}{t^2}\left(\int_0^{\frac{1}{\sqrt{\delta}}}+\int_{\frac{1}{\sqrt{\delta}}}^{\infty}\right)|\varphi''(u)|du+\frac{1}{t^2}\frac{K}{\sqrt{\delta}\psi(\sqrt{\delta})}.$$

Since the function $\varphi(u)$ is convex downward on the intervals $\left[0;\frac{1}{\sqrt{\delta}}\right]$ and $\left[\frac{b}{\sqrt{\delta}};\infty\right)$ and is bounded on the segment $\left[\frac{1}{\sqrt{\delta}};\frac{b}{\sqrt{\delta}}\right]$, using the last inequality we get

$$\left|\int_0^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|\le\frac{1}{t^2}\left(\int_0^{\frac{1}{\sqrt{\delta}}}+\int_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}}+\int_{\frac{b}{\sqrt{\delta}}}^{\infty}\right)|\varphi''(u)|du+\frac{1}{t^2}\frac{K}{\sqrt{\delta}\psi(\sqrt{\delta})}$$

$$=\frac{1}{t^2}\left(\left(\int_0^{\frac{1}{\sqrt{\delta}}}+\int_{\frac{b}{\sqrt{\delta}}}^{\infty}\right)\varphi''(u)\,du+\int_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}}|\varphi''(u)|\,du\right)+\frac{1}{t^2}\frac{K}{\sqrt{\delta}\psi(\sqrt{\delta})}$$

$$\le\frac{1}{t^2}\frac{2\psi(1)-2b\psi(b)-b^2\psi'(b)}{\sqrt{\delta}\psi(\sqrt{\delta})}+\frac{1}{t^2}\frac{K}{\sqrt{\delta}\psi(\sqrt{\delta})}$$

$$+\frac{1}{t^2}\frac{1}{\psi(\sqrt{\delta})}\int_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}}\left(2\psi(\sqrt{\delta}u)+4u\sqrt{\delta}|\psi'(\sqrt{\delta}u)|+u^2\delta\psi''(\sqrt{\delta}u)\right)du$$

$$\le\frac{1}{t^2}\frac{K_1}{\sqrt{\delta}\psi(\sqrt{\delta})}.$$

Then

$$\int_{|t|\ge\sqrt{\delta}}\left|\int_0^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt=O\left(\frac{1}{\delta\psi(\sqrt{\delta})}\right),\quad\delta\to\infty.\qquad(64)$$

Using relation (62), the fact that the function $u^2\psi(u)$ decreases on $[b,\infty)$ and is bounded on $[1,b]$, and inequality (4.16) in [12, p. 59], we obtain

$$\int\limits_0^{\sqrt{\delta}} \left| \int\limits_0^{\infty} \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt$$

$$= \int\limits_0^{\sqrt{\delta}} \left| \left(\int\limits_0^{\frac{1}{\sqrt{\delta}}} + \int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} + \int\limits_{\frac{b}{\sqrt{\delta}}}^{\infty} \right) \varphi(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt$$

$$\leq \sqrt{\delta} \left(\int\limits_0^{\frac{1}{\sqrt{\delta}}} + \int\limits_{\frac{1}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}} \right) \left|\varphi(u)\right| du + \int\limits_0^{\sqrt{\delta}} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+\frac{2\pi}{t}} \frac{u^2\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})} du dt$$

$$\leq \frac{\sqrt{\delta}\psi(1)}{\psi(\sqrt{\delta})} \int\limits_0^{\frac{1}{\sqrt{\delta}}} u^2 du + \frac{1}{\delta\psi(\sqrt{\delta})} \int\limits_1^b u^2 \psi(u)\, du + \frac{1}{\psi(\sqrt{\delta})} \int\limits_0^{\sqrt{\delta}} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+\frac{2\pi}{t}} u^2\psi(\sqrt{\delta}u) du dt$$

$$\leq \frac{K}{\delta\psi(\sqrt{\delta})} + \frac{1}{\psi(\sqrt{\delta})} \int\limits_0^{\sqrt{\delta}} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+\frac{2\pi}{t}} u^2\psi(\sqrt{\delta}u) du dt. \tag{65}$$

Changing variables and integrating by parts in the last integral in (65), we get

$$\frac{1}{\psi(\sqrt{\delta})} \int\limits_0^{\sqrt{\delta}} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+\frac{2\pi}{t}} u^2\psi(\sqrt{\delta}u) du dt$$

$$= \frac{2\pi}{\psi(\sqrt{\delta})} \int\limits_{\frac{2\pi}{\sqrt{\delta}}}^{\infty} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+x} u^2\psi(\sqrt{\delta}u) du \frac{dx}{x^2}$$

$$= \frac{2\pi}{\psi(\sqrt{\delta})} \left(-\frac{1}{x} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+x} u^2\psi(\sqrt{\delta}u) du \Bigg|_{\frac{2\pi}{\sqrt{\delta}}}^{\infty} + \int\limits_{\frac{2\pi}{\sqrt{\delta}}}^{\infty} \frac{1}{x} \left(\frac{b}{\sqrt{\delta}} + x\right)^2 \psi\left(b + \sqrt{\delta}x\right) dx \right)$$

$$= \frac{2\pi}{\psi(\sqrt{\delta})} \left(-\lim_{x\to\infty} \frac{1}{x} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+x} u^2\psi(\sqrt{\delta}u)du + \frac{\sqrt{\delta}}{2\pi} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{(b+2\pi)}{\sqrt{\delta}}} u^2\psi(\sqrt{\delta}u)du \right.$$

$$\left. + \frac{1}{\delta} \int\limits_{\frac{2\pi}{\sqrt{\delta}}}^{\infty} \frac{1}{x} \left(b+\sqrt{\delta}x\right)^2 \psi\left(b+\sqrt{\delta}x\right) dx \right). \tag{66}$$

In the case where

$$\lim_{x\to\infty} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+x} u^2\psi(\sqrt{\delta}u)du = K > 0,$$

we have

$$\lim_{x\to\infty} \frac{1}{x} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+x} u^2\psi(\sqrt{\delta}u)du = 0. \tag{67}$$

In the case where

$$\lim_{x\to\infty} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+x} u^2\psi(\sqrt{\delta}u)du = \infty,$$

using the l'Hospital rule and the fact that

$$\lim_{u\to\infty} u^2\psi(u) = 0$$

we get

$$\lim_{x\to\infty} \frac{1}{x} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{b}{\sqrt{\delta}}+x} u^2\psi(\sqrt{\delta}u)du = \lim_{x\to\infty} \left(\frac{b}{\sqrt{\delta}}+x\right)^2 \psi\left(b+\sqrt{\delta}x\right) = 0. \tag{68}$$

Since the function $\psi(u)$ decreases for $u \geq 1,$ we have

$$\frac{\sqrt{\delta}}{2\pi} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{(b+2\pi)}{\sqrt{\delta}}} u^2\psi(\sqrt{\delta}u)du \leq \frac{\psi(1)\sqrt{\delta}}{2\pi} \int\limits_{\frac{b}{\sqrt{\delta}}}^{\frac{(b+2\pi)}{\sqrt{\delta}}} u^2du \leq \frac{K}{\delta}. \tag{69}$$

By virtue of the summability of the function $u\psi(u)$ on $[1;\infty)$, we get

$$\frac{1}{\delta}\int\limits_{\frac{2\pi}{\sqrt{\delta}}}^{\infty}\frac{1}{x}\left(b+\sqrt{\delta}x\right)^2\psi\left(b+\sqrt{\delta}x\right)dx$$

$$=\frac{1}{\delta}\int\limits_{b+2\pi}^{\infty}\frac{y^2\psi(y)}{y-b}dy=\frac{1}{\delta}\int\limits_{b+2\pi}^{\infty}y\psi(y)\left(1+\frac{b}{y-b}\right)dy$$

$$\leq\frac{\left(1+\dfrac{b}{2\pi}\right)}{\delta}\int\limits_{b+2\pi}^{\infty}y\psi(y)\,dy\leq\frac{K_1}{\delta}. \qquad (70)$$

Using relations (66)–(70) and (65), we obtain

$$\int\limits_{0}^{\sqrt{\delta}}\left|\int\limits_{0}^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt=O\left(\frac{1}{\delta\psi(\sqrt{\delta})}\right),\quad \delta\to\infty. \qquad (71)$$

By analogy, one can show that

$$\int\limits_{-\sqrt{\delta}}^{0}\left|\int\limits_{0}^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt=O\left(\frac{1}{\delta\psi(\sqrt{\delta})}\right),\quad \delta\to\infty. \qquad (72)$$

Using relations (64), (71), and (72), we get

$$A(\varphi)=O\left(\frac{1}{\delta\psi(\sqrt{\delta})}\right),\quad \delta\to\infty.$$

Let us establish the convergence of the integral

$$A(\mu)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left|\int\limits_{0}^{\infty}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt.$$

Integrating twice by parts and taking into account that $\mu(0)=\mu'(0)=0$ and

$$\lim_{u\to\infty}\mu(u)=\lim_{u\to\infty}\mu'(u)=0,$$

we obtain

$$\int\limits_0^{\infty}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du=\left(\int\limits_0^{\frac{1}{\sqrt{\delta}}}+\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}\right)\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du$$

$$=-\frac{1}{t^2}\left(\int\limits_0^{\frac{1}{\sqrt{\delta}}}+\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}\right)\mu''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du$$

$$-\ \frac{1}{t^2}\left(1-e^{-\frac{1}{\delta}}-\frac{1}{\delta}\right)\frac{\sqrt{\delta}\psi'(1)}{\psi(\sqrt{\delta})}\cos\left(\frac{t}{\sqrt{\delta}}+\frac{\beta\pi}{2}\right),$$

whence

$$\left|\int\limits_0^{\infty}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|\le\frac{1}{t^2}\left(\int\limits_0^{\frac{1}{\sqrt{\delta}}}+\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}\right)\big|\mu''(u)\big|\,du+\frac{1}{t^2}\frac{K}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}. \tag{73}$$

For $u\in\left[0,\frac{1}{\sqrt{\delta}}\right]$, we have $\mu''(u)<0$, Hence,

$$\int\limits_0^{\frac{1}{\sqrt{\delta}}}\big|\mu''(u)\big|\,du=-\int\limits_0^{\frac{1}{\sqrt{\delta}}}\mu''(u)du=-\mu'\left(\frac{1}{\sqrt{\delta}}\right)+\mu'(0)\le\frac{K}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}. \tag{74}$$

For $u\ge\frac{1}{\sqrt{\delta}}$, we get

$$\mu''(u)=\left(1-e^{-u^2}-u^2\right)\frac{\delta\psi''(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}+4u\left(e^{-u^2}-1\right)\frac{\sqrt{\delta}\psi'(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}$$

$$+\ 2\left(e^{-u^2}-2u^2e^{-u^2}-1\right)\frac{\psi(\sqrt{\delta}u)}{\psi(\sqrt{\delta})}, \tag{75}$$

whence

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}\big|\mu''(u)\big|\,du\le\frac{\delta}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}\left(e^{-u^2}+u^2-1\right)\psi''(\sqrt{\delta}u)du+\frac{4\sqrt{\delta}}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}u\left(1-e^{-u^2}\right)\Big|\psi'(\sqrt{\delta}u)\Big|\,du$$

$$+\ \frac{2}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}\left(2u^2e^{-u^2}-e^{-u^2}+1\right)\psi(\sqrt{\delta}u)du.$$

Integrating the first and the second integral on the right-hand side of the last inequality twice and once, respectively, by parts, we get

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty} \left|\mu''(u)\right| du \leq -\frac{\sqrt{\delta}\psi'(1)}{\psi(\sqrt{\delta})}\left(e^{-\frac{1}{\delta}}+\frac{1}{\delta}-1\right)+\frac{6\psi(1)}{\sqrt{\delta}\psi(\sqrt{\delta})}\left(1-e^{-\frac{1}{\delta}}\right)$$

$$+\frac{8}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}\left(2u^2e^{-u^2}-e^{-u^2}+1\right)\psi\left(\sqrt{\delta}u\right)du.$$

Using the first and the second inequality from (23), we obtain

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty} \left|\mu''(u)\right| du \leq \frac{K_1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}+\frac{K_2}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty}\left(2u^2e^{-u^2}-e^{-u^2}+1\right)\psi(\sqrt{\delta}u)du. \tag{76}$$

Let us estimate the last integral in inequality (76). To this end, we divide the interval of integration $\left[\frac{1}{\sqrt{\delta}};\infty\right)$ into the two parts $\left[\frac{1}{\sqrt{\delta}};1\right]$ and $[1;\infty)$. Using the third inequality from (23), we get

$$\frac{1}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{1}\left(2u^2e^{-u^2}-e^{-u^2}+1\right)\psi(\sqrt{\delta}u)du \leq \frac{1}{\psi(\sqrt{\delta})}\int\limits_{\frac{1}{\sqrt{\delta}}}^{1}u^2\psi(\sqrt{\delta}u)du=\frac{1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}\int\limits_{1}^{\sqrt{\delta}}u^2\psi(u)du. \tag{77}$$

Taking into account the inequality $2u^2e^{-u^2}-e^{-u^2}+1\leq 2,\ u\in R,$ we obtain

$$\frac{1}{\psi(\sqrt{\delta})}\int\limits_{1}^{\infty}\left(2u^2e^{-u^2}-e^{-u^2}+1\right)\psi(\sqrt{\delta}u)du \leq \frac{2}{\psi(\sqrt{\delta})}\int\limits_{1}^{\infty}\psi\left(\sqrt{\delta}u\right)du\leq K. \tag{78}$$

Using relations (76)–(78), we get

$$\int\limits_{\frac{1}{\sqrt{\delta}}}^{\infty} \left|\mu''(u)\right| du \leq K+\frac{K_1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}+\frac{K_2}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}\int\limits_{1}^{\sqrt{\delta}}u^2\psi(u)du. \tag{79}$$

Combining relations (73), (74), and (79) and taking into account that

$$\lim_{\delta\to\infty}\frac{1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}\int\limits_{1}^{\sqrt{\delta}}u^2\psi(u)du \geq \lim_{\delta\to\infty}\frac{1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}\delta\psi(\sqrt{\delta})\int\limits_{1}^{\sqrt{\delta}}du=1, \tag{80}$$

we obtain

$$\int\limits_{|t|\geq\pi} \left| \int\limits_0^\infty \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt = O\left(\frac{1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})} \int\limits_1^{\sqrt{\delta}} u^2 \psi(u) du \right). \tag{81}$$

Now consider

$$\int\limits_0^\pi \left| \int\limits_0^\infty \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt \leq \int\limits_0^\pi \left| \left(\int\limits_0^{\frac{1}{\sqrt{\delta}}} + \int\limits_{\frac{1}{\sqrt{\delta}}}^1 + \int\limits_1^\infty \right) \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt. \tag{82}$$

Using the inequality

$$e^{-u^2} + u^2 - 1 \leq u^2, \quad u \in R, \tag{83}$$

we get

$$\int\limits_0^\pi \left| \int\limits_0^{\frac{1}{\sqrt{\delta}}} \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt \leq \int\limits_0^\pi \int\limits_0^{\frac{1}{\sqrt{\delta}}} |\mu(u)|\, du dt$$

$$= \frac{\pi\psi(1)}{\psi(\sqrt{\delta})} \int\limits_0^{\frac{1}{\sqrt{\delta}}} \left(e^{-u^2} + u^2 - 1 \right) du \leq \frac{\pi\psi(1)}{\psi(\sqrt{\delta})} \int\limits_0^{\frac{1}{\sqrt{\delta}}} u^2 du \leq \frac{K}{\delta\sqrt{\delta}\psi(\sqrt{\delta})}, \tag{84}$$

$$\int\limits_0^\pi \left| \int\limits_{\frac{1}{\sqrt{\delta}}}^1 \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt \leq \int\limits_0^\pi \int\limits_{\frac{1}{\sqrt{\delta}}}^1 |\mu(u)| du dt$$

$$\leq \frac{\pi}{\psi(\sqrt{\delta})} \int\limits_{\frac{1}{\sqrt{\delta}}}^1 u^2 \psi(\sqrt{\delta}u) du = \frac{\pi}{\delta\sqrt{\delta}\psi(\sqrt{\delta})} \int\limits_1^{\sqrt{\delta}} u^2 \psi(u) du. \tag{85}$$

Since $\psi \in G,$ it is easy to verify that the function $-\mu(u) = (e^{-u^2} + u^2 - 1)\psi(\sqrt{\delta}u)$ is monotonically decreasing beginning with a certain value $u_1 \geq 1.$

Since the function $-\mu(u)$ decreases monotonically on $[u_1; \infty)$, $u_1 \geq 1,$ is nonnegative, and tends to zero as $u \to \infty,$ we can use inequality (4.16) from [12, p. 59]. As a result, we obtain

$$\int_0^\pi \left| \int_1^\infty \mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du \right| dt = \int_0^\pi \left| \int_1^\infty (-\mu(u))\cos\left(ut+\frac{\beta\pi}{2}\right)du \right| dt$$

$$\leq \int_0^\pi \left| \int_1^{u_1} (-\mu(u))\cos\left(ut+\frac{\beta\pi}{2}\right)du \right| dt$$

$$+ \int_0^\pi \left| \int_{u_1}^\infty (-\mu(u))\cos\left(ut+\frac{\beta\pi}{2}\right)du \right| dt$$

$$\leq \int_0^\pi \int_1^{u_1} (-\mu(u))\; du\, dt + \int_0^\pi \int_{u_1}^{u_1+\frac{2\pi}{t}} (-\mu(u))\; dudt$$

$$= \int_0^\pi \int_1^{u_1+\frac{2\pi}{t}} (-\mu(u))\; dudt. \tag{86}$$

Using inequality (83), we get

$$\int_0^\pi \int_1^{u_1+\frac{2\pi}{t}} (-\mu(u))\, dudt \leq \frac{1}{\psi(\sqrt{\delta})} \int_0^\pi \int_1^{u_1+\frac{2\pi}{t}} u^2\psi(\sqrt{\delta}u)dudt. \tag{87}$$

Changing variables and integrating by parts, we obtain

$$\frac{1}{\psi(\sqrt{\delta})} \int_0^\pi \int_1^{u_1+\frac{2\pi}{t}} u^2\psi(\sqrt{\delta}u)dudt = \frac{2\pi}{\psi(\sqrt{\delta})} \int_2^\infty \int_1^{u_1+x} u^2\psi\left(\sqrt{\delta}u\right)du\frac{dx}{x^2}$$

$$= \frac{2\pi}{\psi(\sqrt{\delta})} \left(-\lim_{x\to\infty}\frac{1}{x} \int_1^{u_1+x} u^2\psi(\sqrt{\delta}u)du + \frac{1}{2}\int_1^{2+u_1} u^2\psi(\sqrt{\delta}u)du \right.$$

$$\left. + \int_2^\infty \frac{1}{x}\,(u_1+x)^2\,\psi\left(\sqrt{\delta}(u_1+x)\right)dx \right). \tag{88}$$

In the case where

$$\lim_{x\to\infty} \int_1^{u_1+x} u^2\psi(\sqrt{\delta}u)\, du = K > 0,$$

we obtain

$$\lim_{x\to\infty} \frac{1}{x} \int\limits_1^{u_1+x} u^2\psi(\sqrt{\delta}u)du = 0. \tag{89}$$

In the case where

$$\lim_{x\to\infty} \int\limits_1^{u_1+x} u^2\psi(\sqrt{\delta}u)\, du = \infty,$$

using the l'Hospital rule and the fact that

$$\lim_{u\to\infty} u^2\psi(u) = 0,$$

we get

$$\lim_{x\to\infty} \frac{1}{x} \int\limits_1^{u_1+x} u^2\psi(\sqrt{\delta}u)du = \lim_{x\to\infty} (u_1+x)^2\, \psi\left(\sqrt{\delta}(u_1+x)\right) = 0. \tag{90}$$

Since the function $\psi(u)$ decreases for $u \geq 1,$ we have

$$\frac{1}{2} \int\limits_1^{2+u_1} u^2\psi(\sqrt{\delta}u)du \leq \frac{\psi(\sqrt{\delta})}{2} \int\limits_1^{2+u_1} u^2 du \leq K\psi(\sqrt{\delta}). \tag{91}$$

By virtue of the summability of the function $u\psi(u)$ on $[1;\infty),$ we have

$$\begin{aligned}\int\limits_2^{\infty} \frac{1}{x}\,(u_1+x)^2\,\psi\left(\sqrt{\delta}(u_1+x)\right)dx &= \frac{1}{\delta} \int\limits_{\sqrt{\delta}(2+u_1)}^{\infty} \frac{y^2\psi(y)}{y-\sqrt{\delta}u_1}\,dy \\ &= \frac{1}{\delta} \int\limits_{\sqrt{\delta}(2+u_1)}^{\infty} y\psi(y)\left(1+\frac{\sqrt{\delta}u_1}{y-\sqrt{\delta}u_1}\right)dy \\ &\leq \frac{\left(1+\dfrac{u_1}{2}\right)}{\delta} \int\limits_{\sqrt{\delta}(2+u_1)}^{\infty} y\psi(y)dy \leq \frac{K_1}{\delta} \int\limits_{\sqrt{\delta}}^{\infty} y\psi(y)dy.\end{aligned} \tag{92}$$

Taking relations (89)–(92) into account, we obtain the following inequality from (88):

$$\frac{1}{\psi(\sqrt{\delta})} \int\limits_0^{\pi} \int\limits_1^{u_1+\frac{2\pi}{t}} u^2\psi(\sqrt{\delta}u)dudt \leq K_1 + \frac{K_2}{\delta\psi(\sqrt{\delta})} \int\limits_{\sqrt{\delta}}^{\infty} u\psi(u)du. \tag{93}$$

Using (84), (85), (93), and (80), we deduce the following relation from (82):

$$\int\limits_0^\pi \left| \int\limits_0^\infty \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt = O\left(\frac{1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})} \int\limits_1^{\sqrt{\delta}} u^2\psi(u)du + \frac{1}{\delta\psi(\sqrt{\delta})} \int\limits_{\sqrt{\delta}}^\infty u\psi(u)du \right). \tag{94}$$

By analogy, we get

$$\int\limits_{-\pi}^0 \left| \int\limits_0^\infty \mu(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt = O\left(\frac{1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})} \int\limits_1^{\sqrt{\delta}} u^2\psi(u)du + \frac{1}{\delta\psi(\sqrt{\delta})} \int\limits_{\sqrt{\delta}}^\infty u\psi(u)du \right). \tag{95}$$

Combining relations (81), (94), and (95), we obtain

$$A\left(\mu\right) = O\left(\frac{1}{\delta\sqrt{\delta}\psi(\sqrt{\delta})} \int\limits_1^{\sqrt{\delta}} u^2\psi(u)du + \frac{1}{\delta\psi(\sqrt{\delta})} \int\limits_{\sqrt{\delta}}^\infty u\psi(u)du \right). \tag{96}$$

By analogy with [1, p. 183], one can show that

$$f(x) - W_\delta(f,x) = \psi(\sqrt{\delta}) \int\limits_{-\infty}^\infty f_\beta^\psi\left(x + \frac{t}{\sqrt{\delta}}\right) \hat{\tau}_\beta(t)dt.$$

Hence,

$$\begin{aligned}
\mathcal{E}\left(C_{\beta,\infty}^\psi; W_\delta\right)_C &= \sup_{f\in C_{\beta,\infty}^\psi} \left\| \psi(\sqrt{\delta}) \int\limits_{-\infty}^\infty f_\beta^\psi\left(x + \frac{t}{\sqrt{\delta}}\right) \hat{\tau}_\beta(t)dt \right\|_C \\
&= \sup_{f\in C_{\beta,\infty}^\psi} \left\| \psi(\sqrt{\delta}) \int\limits_{-\infty}^\infty f_\beta^\psi\left(x + \frac{t}{\sqrt{\delta}}\right) \left(\hat{\varphi}_\beta(t) + \hat{\mu}_\beta(t)\right) dt \right\|_C \\
&= \sup_{f\in C_{\beta,\infty}^\psi} \left\| \psi(\sqrt{\delta}) \int\limits_{-\infty}^\infty f_\beta^\psi\left(x + \frac{t}{\sqrt{\delta}}\right) \hat{\varphi}_\beta(t)dt \right\|_C + O\left(\psi(\sqrt{\delta})A(\mu)\right).
\end{aligned} \tag{97}$$

By analogy with relation (1.1) in [10], one can show that the Fourier series of the function

$$f_\varphi(x) = \int\limits_{-\infty}^\infty f_\beta^\psi\left(x + \frac{t}{\sqrt{\delta}}\right) \hat{\varphi}_\beta(t)dt$$

has the form

$$S\left[f_{\varphi}\right]=\sum_{k=1}^{\infty}\frac{k^2}{\delta}\frac{1}{\psi(\sqrt{\delta})}\left(a_k\cos kx+b_k\sin kx\right),$$

where a_k and b_k are the Fourier coefficients of the function f. Therefore,

$$\int_{-\infty}^{\infty}f_{\beta}^{\psi}\left(x+\frac{t}{\sqrt{\delta}}\right)\hat{\varphi}_{\beta}(t)dt=\frac{1}{\delta\psi(\sqrt{\delta})}f_0^{(2)}(x), \tag{98}$$

where $f_0^{(2)}(x)$ is the Weyl–Nagy (r,β)-derivative for $r=2$ and $\beta=0$.

Substituting (98) into (97), we get

$$\mathcal{E}\left(C_{\beta,\infty}^{\psi};W_{\delta}\right)_C=\frac{1}{\delta}\sup_{f\in C_{\beta,\infty}^{\psi}}\left\|f_0^{(2)}(x)\right\|_C+O\left(\psi(\sqrt{\delta})A(\mu)\right),\quad \delta\to\infty. \tag{99}$$

Substituting (96) into (99), we obtain equality (61).

Theorem 2 is proved.

The functions

$$\psi(u)=\frac{1}{u^2\ln^{\alpha}(u+K)},\quad K>0,\quad \alpha>1,$$

and

$$\psi(u)=\frac{1}{u^r}\ln^{\alpha}(u+K),\quad \psi(u)=\frac{1}{u^r}\arctan u,\quad \psi(u)=\frac{1}{u^r}(K+e^{-u}),\quad K>0,\ \ r>2,\ \ \alpha\in R,$$

can serve as examples of functions for which Theorem 2 is true.

3. Estimates for Upper Bounds of Approximations of Functions on the Classes $L_{\beta,1}^{\psi}$ by Weierstrass Integrals in the Integral Metric

Since the function $\tau(u)$ defined by (6) is continuous and (as shown in the proof of Theorem 1) its transform $\hat{\tau}_{\beta}(t)$ of the form (12) is summable, one can prove by analogy with Lemma 2 in [10] that the following equality holds as $\delta\to\infty$:

$$\mathcal{E}\left(L_{\beta,1}^{\psi};W_{\delta}\right)_1=\psi(\sqrt{\delta})A(\tau)+O\left(\psi(\sqrt{\delta})\int_{|t|\geq\frac{\sqrt{\delta}\pi}{2}}\left|\hat{\tau}_{\beta}(t)\right|dt\right).$$

Comparing this relation with (13), we arrive at the following theorem:

Theorem 3. *Suppose that* $\psi \in \mathfrak{M}_0' = \mathfrak{M}_0 \cap \mathfrak{M}'$ *and the function* $g(u) = u^2\psi(u)$ *is convex upward or downward on* $[b;\infty)$, $b \geq 1$. *Then the following equality holds as* $\delta \to \infty$:

$$\mathcal{E}\left(L^{\psi}_{\beta,1}; W_\delta\right)_1 = \psi(\sqrt{\delta})A(\tau) + O\left(\frac{1}{\delta} + \frac{\psi(\sqrt{\delta})}{\sqrt{\delta}}\right),$$

where $A(\tau)$ *is defined by (7) and satisfies estimate (15).*

Using Theorem 3 and reasoning by analogy with the proof of Corollaries 1–3, we establish the following statements:

Corollary 4. *If the conditions of Theorem 1 are satisfied,* $\sin\dfrac{\beta\pi}{2} \neq 0$, *and* $\lim\limits_{t\to\infty} \alpha(t) = \infty$, *where* $\alpha(t)$ *is defined by (11), then the following asymptotic equality holds as* $\delta \to \infty$:

$$\mathcal{E}\left(L^{\psi}_{\beta,1}; W_\delta\right)_1 = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \int\limits_{\sqrt{\delta}}^{\infty} \frac{\psi(u)}{u}du + O\left(\psi(\sqrt{\delta})\right). \tag{100}$$

Corollary 5. *Suppose that* $\psi \in \mathfrak{M}_0$, $\sin\dfrac{\beta\pi}{2} \neq 0$, *the function* $u^2\psi(u)$ *is convex upward or downward for* $u \geq b \geq 1$, *and*

$$\lim_{u\to\infty} u^2\psi(u) = \infty, \lim_{\delta\to\infty} \frac{1}{\delta\psi(\sqrt{\delta})} \int\limits_1^{\sqrt{\delta}} u\psi(u)du = \infty.$$

Then the following asymptotic equality holds as $\delta \to \infty$:

$$\mathcal{E}\left(L^{\psi}_{\beta,1}; W_\delta\right)_1 = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta} \int\limits_1^{\sqrt{\delta}} u\psi(u)du + O\left(\psi(\sqrt{\delta})\right). \tag{101}$$

Corollary 6. *Suppose that* $\psi \in \mathfrak{M}_0$, $\sin\dfrac{\beta\pi}{2} \neq 0$, *the function* $u^2\psi(u)$ *is convex downward on* $[b;\infty)$, $b \geq 1$, $\lim\limits_{u\to\infty} u^2\psi(u) = K < \infty$, *and*

$$\lim_{\delta\to\infty} \int\limits_1^{\sqrt{\delta}} u\psi(u)du = \infty.$$

Then the following asymptotic equality holds as $\delta \to \infty$:

$$\mathcal{E}\left(L^{\psi}_{\beta,1}; W_\delta\right)_1 = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta} \int\limits_1^{\sqrt{\delta}} u\psi(u)du + O\left(\frac{1}{\delta}\right). \tag{102}$$

Note that, under the conditions of Corollaries 4–6, equalities (100)–(102) give a solution of the Kolmogorov–Nikol'skii problem for the Weierstrass integrals W_δ on the classes $L^{\psi}_{\beta,1}$ in the integral metric.

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