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ASYMPTOTICS OF THE VALUES OF APPROXIMATIONS IN THE MEAN FOR CLASSES OF DIFFERENTIABLE FUNCTIONS BY USING BIHARMONIC POISSON INTEGRALS

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We obtain complete asymptotic expansions for the exact upper bounds of the approximations of functions from the classes W_1^r, $r \in N$, and $\overline{W}_1^r$, $r \in N \setminus \{1\}$, by their biharmonic Poisson integrals.

Let C be the space of 2π-periodic continuous functions with norm specified by the equality $\|f\|_C = \max_t |f(t)|$, let L_∞ be the space of 2π-periodic measurable essentially bounded functions with norm $\|f\|_\infty = \operatorname{ess\,sup}_t |f(t)|$, and let L be the space of 2π-periodic functions summable over a period with the following norm:

$$\|f\|_L = \|f\|_1 = \int_{-\pi}^{\pi} |f(t)|dt.$$

Further, let W_p^r (where $p = 1$ or $p = \infty$) be the set of 2π-periodic functions with absolutely continuous derivatives up to the $(r-1)$th order, inclusively, such that $\left\| f^{(r)}(t) \right\|_p \le 1$ for $p = 1, \infty$ and let $\overline{W}_p^r$ be the class of functions conjugate to the functions from the class W_p^r, i.e.,

$$\overline{W}_p^r = \left\{ \bar{f} : \bar{f}(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \cot\frac{t}{2} dt, \ f \in W_p^r \right\}, \tag{1}$$

where the integral is understood in the sense of its principal value, i.e.,

$$\int_{-\pi}^{\pi} f(x+t)\cot\frac{t}{2}dt = \lim_{\varepsilon \to 0+} \left(\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \right) f(x+t)\cot\frac{t}{2}dt$$

(see, e.g., [1, p. 22]).

Also let $f \in L$. The quantity

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$$B_\delta(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)K_\delta(t)\,dt, \qquad \delta > 0, \quad -\pi \le x < \pi, \tag{2}$$

is called the biharmonic Poisson integral of the function f, where

$$K_\delta(t) = \frac{1}{2} + \sum_{k=1}^{\infty}\left[1+\frac{k}{2}\left(1-e^{-2/\delta}\right)\right]e^{-k/\delta}\cos kt \tag{3}$$

is the biharmonic Poisson kernel (see [2]).

Further, by B_δ we denote the periodic extension of the function $B_\delta(f,x)$, $x\in[-\pi;\pi)$, onto the entire real axis.

Denote

$$\mathcal{E}(\mathfrak{N}, B_\delta)_1 = \sup_{f\in\mathfrak{N}}\| f(x) - B_\delta(f,x)\|_1, \tag{4}$$

$$\mathcal{E}(\mathfrak{N}, B_\delta)_C = \sup_{f\in\mathfrak{N}}\| f(x) - B_\delta(f,x)\|_C, \tag{5}$$

where $\mathfrak{N} \equiv W_p^r$ or $\mathfrak{N} \equiv \overline{W}_p^r$, $p = 1, \infty$.

If we know the explicit form of a function $g(\delta) = g(\mathfrak{N};\delta)$ such that the following exact asymptotic equality

$$\mathcal{E}(\mathfrak{N}, B_\delta)_X = g(\delta) + o(g(\delta)), \tag{6}$$

holds as $\delta\to\infty$, then, following Stepanets [3, p. 198], we say that the Kolmogorov–Nikol'skii problem is solved for the indicated class $\mathfrak{N}$ and the operator $B_\delta(f,x)$ in the metric of the space X.

A formal series $\sum_{n=0}^{\infty} g_n(\delta)$ is called the *complete asymptotic expansion* or the *complete asymptotics* of the function $f(\delta)$ as $\delta\to\infty$ if

$$|g_{n+1}(\delta)| = o(|g_n(\delta)|) \tag{7}$$

for all $n\in N$ and

$$f(\delta) = \sum_{n=0}^{N} g_n(\delta) + o(g_N(\delta)) \qquad \text{as} \quad \delta\to\infty. \tag{8}$$

for any natural N. We also represent this result in the following brief form: $f(\delta) \cong \sum_{n=0}^{\infty} g_n(\delta)$.

The aim of the present paper is to deduce complete asymptotic expansions of quantities (4) for $\mathfrak{N} = W_1^r$, $r\in N$, and $\mathfrak{N} = \overline{W}_1^r$, $r\in N\setminus\{1\}$, in powers of $\frac{1}{\delta}$ as $\delta\to\infty$.

Theorem 1. *The following asymptotic expansion is true:*

$$\mathcal{E}(W_1^1; B_\delta)_1 \cong \frac{2}{\pi}\left(\frac{1}{\delta} + \sum_{k=2}^{\infty} \nu_k^1 \frac{1}{\delta^k}\right) \quad \text{as} \quad \delta \to \infty, \tag{9}$$

where

$$\nu_k^1 = (-1)^{k-1}\frac{1-k}{k!}\sigma_{k-1}, \quad k = 2, 3, \ldots, \tag{10}$$

$$\sigma_j = \begin{cases} 0, & j = 2l-1, \\ \dfrac{1}{2^{j-1} j!}\displaystyle\sum_{i=1}^{j}(2i-1)^j a_i^{j+1} - \dfrac{2^j (j-1)!}{(2j)!}\displaystyle\sum_{i=0}^{j-1}(-1)^i C_j^i (j-i)^{2j}, & j = 2l, \end{cases} \quad l \in N, \tag{11}$$

$$a_i^j = \begin{cases} 1, & i = 1, \quad i = j-1, \\ a_i^{j-1}(2i-1) + a_{j-i}^{j-1}(2(j-i)-1), & 1 < i < j-1, \end{cases} \quad j \in N. \tag{12}$$

Proof. In [4], we established the complete asymptotic expansion

$$\mathcal{E}(W_\infty^1; B_\delta)_C \cong \frac{2}{\pi}\left(\frac{1}{\delta} + \sum_{k=2}^{\infty} \nu_k^1 \frac{1}{\delta^k}\right) \quad \text{as} \quad \delta \to \infty,$$

where ν_k^1 is given by relation (10). In deducing this expansion, we used the following Falaleev's equality [5, p. 164]:

$$\mathcal{E}(W_\infty^1; B_\delta)_C = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1-\left(1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right)e^{-(2k+1)/\delta}}{(2k+1)^2}. \tag{13}$$

Hence, it is clear that, in order to get relation (9), it suffices to show that $\mathcal{E}(W_1^1; B_\delta)_1$ coincides with the right-hand side of relation (13) or, equivalently, that $\mathcal{E}(W_1^1; B_\delta)_1 = \mathcal{E}(W_\infty^1; B_\delta)_C$.

By using the integral representation (2) and the fact that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_\delta(t)\,dt = 1,$$

we find

$$f(x) - B_\delta(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x) - f(t+x))K_\delta(t)\,dt. \tag{14}$$

Since the function $(f(x) - f(t+x))K_\delta(t)$ is measurable on the set $[-\pi; \pi] \times [-\pi; \pi]$ and

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} \left|\left(f(x) - f(t+x)\right) K_\delta(t)\right| dt \; < \; +\infty,$$

by virtue of the corollary of the Fubini theorem (see, e.g., [6, p. 331]), substituting the right-hand side of equality (14) in relation (4), in view of the facts that

$$\int_{-\pi}^{\pi} |f(x+t) - f(x)| dx \;\le\; |t|$$

for $f \in W_1^1$ and $K_\delta(t) \ge 0$ for $\delta > 0$, $-\pi \le x < \pi$, we conclude that

$$\mathcal{E}(W_1^1; B_\delta)_1 \;\le\; \frac{2}{\pi}\int_0^{\pi} t K_\delta(t)\, dt \;=\; \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1 - \left(1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right) e^{-(2k+1)/\delta}}{(2k+1)^2}. \tag{15}$$

On the other hand, in view of the lemma from [7, p. 63], we get

$$\mathcal{E}(W_1^1; B_\delta)_1 \;\ge\; \sup_{f \in T^1} \int_{-\pi}^{\pi} \left| f(x) - B_\delta(f, x)\right| dx \;\ge\; \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1 - \left(1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right) e^{-(2k+1)/\delta}}{(2k+1)^2}, \tag{16}$$

where T^n is the class of all trigonometric polynomials g such that

$$\int_{-\pi}^{\pi} \left| g^{(n)}(x) \right| dx \;\le\; 1.$$

By using inequalities (15) and (16) and relation (13), we obtain

$$\mathcal{E}(W_1^1; B_\delta)_1 \;=\; \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1 - \left(1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right) e^{-\frac{2k+1}{\delta}}}{(2k+1)^2} \;=\; \mathcal{E}(W_\infty^1; B_\delta)_C. \tag{17}$$

Theorem 1 is thus proved.

Theorem 2. *If* $r = 2l + 1$, $l \in N$, *then the following complete asymptotic expansion is true:*

$$\mathcal{E}(W_1^r; B_\delta)_1 \;\cong\; \frac{2}{\pi}\left(\frac{1-r}{r!}\frac{1}{\delta^r}\ln\delta + \sum_{k=2}^{\infty} \nu_k^r \frac{1}{\delta^k}\right) \quad \textit{as} \quad \delta \to \infty, \tag{18}$$

where

$$\nu_k^r = \begin{cases} \dfrac{(-1)^{k-1}(1-k)}{k!}\varphi_{r-k}(0), & k < r, \\ \dfrac{1}{r!}\left((1-r)\left(\ln 2 + \sum\limits_{i=1}^{r}\dfrac{1}{i}\right)+1\right), & k = r, \\ \dfrac{(-1)^{k-1}(1-k)}{k!}\sigma_{k-r}, & k > r, \quad k = 2, 3, \ldots, \end{cases} \tag{19}$$

σ_j is given by relation (11), and

$$\varphi_n(0) = \begin{cases} \dfrac{\pi}{2}K_n, & n = 2l-1, \\ \dfrac{\pi}{2}\tilde{K}_n, & n = 2l, \end{cases} \qquad l \in N, \tag{20}$$

where K_n and $\tilde{K}_n$ are the well-known Favard–Akhiezer–Krein constants:

$$K_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m(n+1)}}{(2m+1)^{n+1}}, \qquad n = 0, 1, 2, \ldots,$$

$$\tilde{K}_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{mn}}{(2m+1)^{n+1}}, \qquad n \in N.$$

Proof. In [4] (Theorem 2), we established the following complete asymptotic expansion:

$$\mathcal{E}(W_\infty^r; B_\delta)_C \cong \frac{2}{\pi}\left(\frac{1-r}{r!}\frac{1}{\delta^r}\ln\delta + \sum_{k=2}^{\infty}\nu_k^r\frac{1}{\delta^k}\right), \qquad \delta \to \infty,$$

where ν_k^r are the coefficients given by relations (19).

Thus, to prove the theorem, it suffices to show that the equalities

$$\mathcal{E}(W_1^r; B_\delta)_1 = \mathcal{E}(W_\infty^r; B_\delta)_C, \qquad r = 2l+1, \qquad l \in N, \tag{21}$$

are true in view of the fact that, according to relation (47) in [4],

$$\mathcal{E}(W_\infty^r; B_\delta)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1 - \left(1 + \dfrac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right)e^{-(2k+1)/\delta}}{(2k+1)^{r+1}}. \tag{22}$$

As a result of the r-fold integration of relation (14) by parts, we find

$$f(x) - B_\delta(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(x+t)Q_r(t;\delta)\,dt,$$

where

$$Q_r(t;\delta) = \sum_{k=1}^{\infty} \frac{1-\left(1+\frac{k}{2}\left(1-e^{-2/\delta}\right)\right)e^{-k/\delta}}{k^r} \cos\left(kt+\frac{r\pi}{2}\right). \tag{23}$$

Therefore,

$$\mathcal{E}(W_1^r; B_\delta)_1 = \sup_{f \in W_1^r} \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f^{(r)}(t+x) Q_r(t;\delta)\, dt \right| dx. \tag{24}$$

For the subsequent evaluation of the quantity $\mathcal{E}(W_1^r; B_\delta)_1$, we first show that

$$\operatorname{sgn} Q_r(t;\delta) = \pm \operatorname{sgn} \sin t, \qquad r = 2l+1. \tag{25}$$

It is clear that, for $r = 2l+1$, $l \in N$, we have

$$Q_r(0;\delta) = Q_r(\pi;\delta) = 0.$$

Under the assumption that $Q_r(t;\delta)$ is equal to zero at a certain additional point $t_0 \in (0, \pi)$, by the Rolle theorem, one can find points $t_1^{(1)} \in (0, t_0)$ and $t_1^{(2)} \in (t_0, \pi)$ such that

$$Q_r'(t_1^{(1)};\delta) = Q_r'(t_1^{(2)};\delta) = 0.$$

This yields

$$Q_{r-1}(t_1^{(1)};\delta) = Q_{r-1}(t_1^{(2)};\delta) = 0$$

and, hence, there exists a point $t_2 \in \left(t_1^{(1)}, t_1^{(2)}\right)$ such that

$$Q_{r-2}(t_2;\delta) = 0,$$

etc. Further, we perform the outlined procedure $r-2$ times and, as a result, conclude that there exist points $t_{r-2}^{(1)} \in (0, t_{r-1})$ and $t_{r-2}^{(2)} \in (t_{r-1}, \pi)$ such that

$$Q_2\left(t_{r-2}^{(1)};\delta\right) = Q_2\left(t_{r-2}^{(2)};\delta\right) = 0.$$

which contradicts the fact that the function $Q_2(t;\delta)$ is equal to zero at a single point inside the interval $(0;\pi)$. Indeed,

$$Q_2'(t;\delta) = -\sum_{k=1}^{\infty} \frac{\sin kt}{k} + \sum_{k=1}^{\infty} \frac{e^{-k/\delta} \sin kt}{k} + \frac{1}{2}\left(1-e^{-2/\delta}\right) \sum_{k=1}^{\infty} e^{-\frac{k}{\delta}} \sin kt.$$

By using relations (1.441.1), (1.447.1), and (1.448.1) from [8], conclude that

$$Q_2'(t;\delta) = \frac{t-\pi}{2} + \arctan\frac{e^{-1/\delta}\sin t}{1-e^{-1/\delta}\cos t} + \frac{\left(1-e^{-2/\delta}\right)e^{-1/\delta}\sin t}{2\left(1-2e^{-1/\delta}\cos t+e^{-2/\delta}\right)}.$$

Further, we get

$$Q_2''(t;\delta) = \frac{\left(e^{-2/\delta}-1\right)^2\left(1-e^{-1/\delta}\cos t\right)}{2\left(1-2e^{-1/\delta}\cos t+e^{-2/\delta}\right)^2}$$

and it is easy to see that $Q_2''(t;\delta) > 0$, $t\in(0;\pi)$. Thus, $Q_2'(t;\delta)$ increases on $(0;\pi)$. Moreover, since $Q_2'(0;\delta) = -\frac{\pi}{2}$ and $Q_2'(\pi;\delta)=0$, we have $Q_2'(t;\delta)<0$ on $(0;\pi)$. Therefore, $Q_2(t;\delta)$ decreases on $(0;\pi)$ and, in view of the fact that $Q_2(0;\delta) > 0$ and $Q_2(\pi;\delta) < 0$, we conclude that the function $Q_2(t;\delta)$ is equal to zero at a single point of the interval $(0;\pi)$.

Equality (25) is proved. Thus, by using relation (24) with $r = 2l+1$, $l\in N$, we obtain

$$\mathscr{E}(W_1^r;B_\delta)_1 \le \frac{1}{\pi}\int_{-\pi}^{\pi}\left|Q_r(t;\delta)\right|dt = \frac{2}{\pi}\left|\int_0^{\pi}\sum_{k=1}^{\infty}\frac{1-\left[1+\frac{k}{2}\left(1-e^{-2/\delta}\right)\right]e^{-k/\delta}}{k^r}\sin kt\,dt\right|$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{26}$$

On the other hand, in view of the lemma from [7, p. 63], for odd r, we get

$$\mathscr{E}(W_1^r;B_\delta)_1 \ge \sup_{f\in T^r}\int_{-\pi}^{\pi}\left|f(x)-B_\delta(f,x)\right|dx \ge \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{27}$$

Comparing relations (26) and (27), we conclude that

$$\mathscr{E}(W_1^r;B_\delta)_1 = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}.$$

By using relation (22), we arrive at relation (21) and, hence, at relation (18).

Theorem 2 is thus proved.

Theorem 3. *If* $r=2l$, $l\in N$, *then the following complete asymptotic expansion is true as* $\delta\to\infty$:

$$\mathcal{E}(W_1^r; B_\delta)_1 \cong \frac{4}{\pi}\sum_{k=2}^{\infty}\eta_k^r \frac{1}{\delta^k}, \tag{28}$$

where

$$\eta_k^r = \begin{cases} \dfrac{(-1)^{k-1}(1-k)}{k!}\psi_{r-k}(0), & k<r, \\ \dfrac{r-1}{r!}\dfrac{\pi}{4}, & k=r, \\ \dfrac{(1-k)}{k!}\tau_{k-r}, & k>r, \quad k=2,3,\ldots, \end{cases} \tag{29}$$

$$\tau_j = \begin{cases} 0, & j=2l, \\ \dfrac{1}{2^j}\sum_{i=1}^{j}(-1)^{i-1}a_i^{j+1}, & j=2l-1, \end{cases} \qquad l\in N, \tag{30}$$

the coefficients a_i^j *are given by relation (12), and*

$$\psi_n(0) = \begin{cases} \dfrac{\pi}{4}\tilde{K}_n, & n=2l-1, \\ \dfrac{\pi}{4}K_n, & n=2l, \end{cases} \qquad l\in N. \tag{31}$$

Proof. By virtue of Theorem 3 in [4], the following complete asymptotic expansion is true:

$$\mathcal{E}(W_\infty^r; B_\delta)_C \cong \frac{4}{\pi}\sum_{k=1}^{\infty}\eta_k^r\frac{1}{\delta^k} \quad \text{as} \quad \delta\to\infty,$$

where the coefficients η_k^r are given by relation (29). According to relation (50) in [4], we also have

$$\mathcal{E}(W_\infty^r; B_\delta)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\,\frac{1-\left(1+\dfrac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right)e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{32}$$

Therefore, to prove the theorem, it suffices to show that

$$\mathcal{E}(W_1^r; B_\delta)_1 = \mathcal{E}(W_\infty^r; B_\delta)_C, \qquad r=2l, \qquad l\in N,$$

or, equivalently, that $\mathcal{E}(W_1^r; B_\delta)_1$ coincides with right-hand side of relation (32).

As shown in the proof of Theorem 1, equality (24) is true. Let us show that

$$\operatorname{sgn}\left(Q_r(t;\delta)-Q_r\left(\frac{\pi}{2};\delta\right)\right) = \pm\operatorname{sgn}\cos t, \qquad r=2l, \qquad l\in N. \tag{33}$$

For $r = 2$, the validity of equality (33) follows from the fact the function $Q_2(t;\delta)$ possesses a single zero on $(0;\pi)$.

We now show that equality (33) is true for $r = 2l + 2$, $l \in N$. Assume that

$$Q_r(t_0;\delta) - Q_r\left(\frac{\pi}{2};\delta\right) = 0, \qquad t_0 \in (0,\pi), \qquad t_0 \neq \frac{\pi}{2}.$$

Then, according to the Rolle theorem, there exists a point $t_1 \in (0,\pi)$ such that

$$Q_r'(t_1;\delta) = 0,$$

whence it follows that

$$Q_{r-1}(t_1;\delta) = 0,$$

which is impossible in view of relation (25). Equality (33) is proved. Thus, by using relation (24) and the corollary of the Fubini theorem [5, p. 331] whose conditions are clearly satisfied for $r = 2l$, $l \in N$, we find

$$\begin{aligned}
\mathcal{E}(W_1^r;B_\delta)_1 &= \sup_{f\in W_1^r}\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} f^{(r)}(x+t)\left(Q_r(t;\delta) - Q_r\left(\frac{\pi}{2};\delta\right)\right)dt\right|dx \\
&\le \frac{1}{\pi}\int_{-\pi}^{\pi}\left|Q_r(t;\delta) - Q_r\left(\frac{\pi}{2};\delta\right)\right|dt = \frac{2}{\pi}\left|\left(\int_0^{\pi/2} - \int_{\pi/2}^{\pi}\right)\left(Q_r(t;\delta) - Q_r\left(\frac{\pi}{2};\delta\right)\right)dt\right| \\
&= \frac{4}{\pi}\left|\int_0^{\pi/2}\sum_{k=0}^{\infty}\frac{1-\left(1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right)e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}\cos(2k+1)t\,dt\right| \\
&= \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1-\left(1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right)e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}.
\end{aligned} \tag{34}$$

On the other hand, by virtue of the lemma in [7, p. 63], for even r, we have

$$\mathcal{E}(W_1^r;B_\delta)_1 \ge \sup_{f\in T^r}\int_{-\pi}^{\pi}\left|f(x) - B_\delta(f,x)\right|dx \ge \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1-\left(1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right)e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{35}$$

Thus, in view of relations (34), (35), and (32), we arrive at the equality

$$\mathcal{E}(W_1^r; B_\delta)_1 = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{1-\left(1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right)e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}} = \mathcal{E}(W_\infty^r; B_\delta)_C.$$

Theorem 3 is proved.

The complete asymptotic expansions for approximations from the classes $\overline{W}_1^r$ are presented in Theorems 4 and 5.

Theorem 4. *If* $r = 2l$, $l \in N$, *then the following complete asymptotic expansion is true as* $\delta \to \infty$:

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 \cong \frac{2}{\pi}\left(\frac{r-1}{r!}\frac{1}{\delta^r}\ln\delta + \sum_{k=2}^{\infty}\overline{\nu}_k^r\frac{1}{\delta^r}\right), \tag{36}$$

where $\overline{\nu}_k^r = \nu_k^r$ *for* $k \neq r$, $\overline{\nu}_r^r = -\overline{\nu}_r^r$ *and the coefficients* ν_k^r, $k = 2, 3, \ldots$, *are given by relation (19).*

Proof. The following complete asymptotic expansion is obtained in Theorem 4 from [4]:

$$\mathcal{E}(\overline{W}_\infty^r; B_\delta)_C \cong \frac{2}{\pi}\left(\frac{r-1}{r!}\frac{1}{\delta^r}\ln\delta + \sum_{k=2}^{\infty}\overline{\nu}_k^r\frac{1}{\delta^r}\right), \qquad \delta \to \infty.$$

As earlier, to prove this theorem, it suffices to show that

$$\mathcal{E}(\overline{W}_\infty^r; B_\delta)_C = \mathcal{E}(\overline{W}_1^r; B_\delta)_1, \qquad r = 2l, \quad l \in N,$$

provided that the equality

$$\mathcal{E}(\overline{W}_\infty^r; B_\delta)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-\left(1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right)e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}} \tag{37}$$

holds for $\mathcal{E}(\overline{W}_\infty^r; B_\delta)_C$, $r = 2l$, $l \in N$ (see [4, p. 23]).

By using the integral representation (1) and the fact that

$$\overline{B}_\delta(f, x) = B_\delta(\bar{f}, x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\sum_{k=1}^{\infty}\left[1+\frac{k}{2}\left(1-e^{-2/\delta}\right)\right]e^{-k/\delta}\sin kt\,dt,$$

for $f \in W^r$, $r \in N \setminus \{1\}$, and integrating by parts r times, we get

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 = \frac{1}{\pi}\sup_{f\in W^r}\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} f^{(r)}(t+x)\overline{Q}_r(t;\delta)\,dt\right|dx, \tag{38}$$

where

$$\overline{Q}_r(t;\delta) = \sum_{k=1}^{\infty} \frac{1-\left[1+\frac{k}{2}\left(1-e^{-2/\delta}\right)\right]e^{-k/\delta}}{k^r}\cos\left(kt+\frac{(r+1)\pi}{2}\right), \qquad \delta>0.$$

We now show that

$$\operatorname{sgn}\overline{Q}_r(t;\delta) = \pm\operatorname{sgn}\sin t, \qquad r=2l, \quad l\in N. \tag{39}$$

Clearly,

$$\overline{Q}_r(0;\delta) = \overline{Q}_r(\pi;\delta) = 0, \qquad r=2l, \qquad l\in N.$$

We assume that

$$\overline{Q}_r(t;\delta) = 0$$

for some additional $t_0\in(0,\pi)$. By applying the Rolle theorem $r-2$ times, we conclude that, for the function $\overline{Q}_2(t;\delta)$, there exists $t_{r-2}\in(0,\pi)$ such that

$$\overline{Q}_2(t_{r-2};\delta) = 0,$$

which is impossible because, by virtue of the remark to Theorem 1.14 in [9, p. 297],

$$\overline{Q}_2(t;\delta) > 0, \quad t\in(0,\pi).$$

Therefore, equality (39) is true.

Hence, it follows from relation (38) with $r=2l$, $l\in N$, that

$$\mathscr{E}(\overline{W}_1^r;B_\delta)_1 \le \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\overline{Q}_r(t;,\delta)\right|dt = \frac{2}{\pi}\left|\int_0^{\pi}\sum_{k=1}^{\infty}\frac{1-\left[1+\frac{k}{2}\left(1-e^{-2/\delta}\right)\right]e^{-k/\delta}}{k^r}\sin kt\,dt\right|$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{40}$$

On the other hand, by using the lemma from [7, p. 63], for even r, we obtain

$$\mathscr{E}(\overline{W}_1^r;B_\delta)_1 \ge \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{41}$$

Comparing relations (40) and (41), in view of equality (37), we find

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}} = \mathcal{E}(\overline{W}_\infty^r; B_\delta)_C.$$

Theorem 4 is thus proved.

Theorem 5. *If* $r = 2l+1$, $l \in N$, *then the following complete asymptotic expansion is true as* $\delta \to \infty$:

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 \cong \frac{4}{\pi}\sum_{k=2}^{\infty}\overline{\eta}_k^r\frac{1}{\delta^k}, \tag{42}$$

where $\overline{\eta}_k^r = \eta_k^r$ *for* $k \neq r$, $\overline{\eta}_r^r = -\overline{\eta}_r^r$, *and the coefficients* η_k^r, $k = 2, 3, \ldots$, *are given by equality (29).*

Proof. By virtue virtue of Theorem 5 in [4], we have the following complete asymptotic expansion:

$$\mathcal{E}(\overline{W}_\infty^r; B_\delta)_C \cong \frac{4}{\pi}\sum_{k=2}^{\infty}\overline{\eta}_k^r\frac{1}{\delta^k} \quad \text{as} \quad \delta \to \infty.$$

Therefore, to prove the theorem, it suffices to check the equality

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 = \mathcal{E}(\overline{W}_\infty^r; B_\delta)_C, \qquad r = 2l+1, \qquad l \in N, \tag{43}$$

by using the fact that, according to relation (57) in [4], we have

$$\mathcal{E}(\overline{W}_\infty^r; B_\delta)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1-\left[1+\frac{2k+1}{2}\left(1-e^{-2/\delta}\right)\right]e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{44}$$

As shown in the proof of Theorem 4, equality (38) is true. We now demonstrate that

$$\operatorname{sgn}\left(\overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right) = \pm\operatorname{sgn}\cos t, \qquad r = 2l+1, \quad l \in N. \tag{45}$$

To do this, we assume that

$$\overline{Q}_r(t_0;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right) = 0, \qquad t_0 \in (0,\pi), \qquad t_0 \neq \frac{\pi}{2}.$$

Thus, according to the Rolle theorem, there exists a point $t_1 \in (0, \pi)$ such that

$$\overline{Q}_r'(t_1;\delta) = 0,$$

whence it follows that

$$\overline{Q}_{r-1}(t_1;\delta) = 0,$$

However, by virtue of relation (39), this is impossible. Equality (45) is proved. Thus, by using relation (38) and the corollary of the Fubini theorem [6, p. 331], for $r = 2l + 1$, $l \in N$, we find

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 = \sup_{f \in W_1^r} \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f^{(r)}(x+t)\left(\overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right)\right) dt \right| dx$$

$$\leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right) \right| dt = \frac{2}{\pi} \left| \left(\int_0^{\pi/2} - \int_{\pi/2}^{\pi} \right) \left(\overline{Q}_r(t;\delta) - \overline{Q}_r\left(\frac{\pi}{2};\delta\right) \right) dt \right|$$

$$= \frac{4}{\pi} \left| \int_0^{\pi/2} \sum_{k=0}^{\infty} \frac{1 - \left[1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}} \cos(2k+1)t \, dt \right|$$

$$= \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1 - \left[1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{46}$$

On the other hand, according to the lemma from [7, p. 63], for odd r, we have

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 \geq \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1 - \left[1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}}. \tag{47}$$

By using relations (46), (47), and (44), we arrive at the equality

$$\mathcal{E}(\overline{W}_1^r; B_\delta)_1 = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1 - \left[1 + \frac{2k+1}{2}\left(1 - e^{-2/\delta}\right)\right] e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}} = \mathcal{E}(\overline{W}_\infty^r; B_\delta)_C.$$

Theorem 5 is proved.

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