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APPROXIMATION OF CLASSES OF PERIODIC MULTIVARIABLE FUNCTIONS BY LINEAR POSITIVE OPERATORS

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In an N-dimensional space, we consider the approximation of classes of translation-invariant periodic functions by a linear operator whose kernel is the product of two kernels one of which is positive. We establish that the least upper bound of this approximation does not exceed the sum of properly chosen least upper bounds in m- and $((N-m))$-dimensional spaces. We also consider the cases where the inequality obtained turns into the equality.

It is known that the determination of least upper bounds for the approximation of classes of periodic functions by linear operators in an N-dimensional space is not always reducible to the calculation of the corresponding quantities in a space of lower dimension. In the present work, we indicate conditions under which this reduction is possible.

In what follows, we prove that the least upper bound of the approximation of classes of translation-invariant periodic functions by a linear operator whose kernel is the product of two kernels one of which is m-dimensional and positive does not exceed the sum of properly chosen least upper bounds in m- and $(N-m)$-dimensional spaces. We establish that the inequality obtained turns into the equality in the case of centrally symmetric classes of continuous or essentially bounded functions that satisfy an additional condition. The results obtained yield, as a consequence, Theorem 1 from [1].

We show that the approximation by a linear positive operator with arbitrary kernel depends only on the one-dimensional components of this kernel, i.e., it coincides with the approximation by a linear positive operator whose kernel is the product of one-dimensional kernels.

Let C^N, L_∞^N, and L_p^N be the spaces of functions $f(x) = f(x_1, \ldots, x_N)$ that are 2π-periodic in each of N variables and are, respectively, continuous, essentially bounded, and summable to the pth power $(1 \le p < \infty)$ with the norms

$$\|f\|_{C^N} = \sup_x |f(x)|, \qquad \|f\|_{L_\infty^N} = \sup_x \operatorname{vrai} |f(x)|,$$

$$\|f\|_{L_p^N} = \left(\left(\frac{1}{2\pi}\right)^N \int_{P_N} |f(x)|^p \, dx \right)^{1/p},$$

where

$$P_N = \prod_{i=1}^{N} [0; 2\pi]$$

is an N-dimensional cube.

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Let $x^m = (x_1, \dots, x_m)$, $x^{N-m} = (x_{m+1}, \dots, x_N)$, and $x = (x_1, \dots, x_N)$ denote, respectively, m-, $(N-m)$-, and N-dimensional vectors whose coordinates are real numbers. Let E^m and E^{N-m} be the sets of all m- and $(N-m)$-dimensional vectors $k^m = (k_1, \dots, k_i, \dots, k_m)$ and $k^{N-m} = (k_{m+1}, \dots, k_N)$ whose coordinates are non-negative integer numbers, let $S(k^m)$ and $S(k^{N-m})$ denote the numbers of the coordinates of k^m and k^{N-m} that are equal to zero, let B^m and B^{N-m} be the sets of all possible m- and $(N-m)$-dimensional vectors each coordinate of which is equal to either 0 or 1, let $B(k^m)$ and $B(k^{N-m})$ denote the subsets of the sets B^m and B^{N-m} such that if $k_j = 0$, then $i_j = 0$, where $i^m = (i_1, \dots, i_j, \dots, i_m) \in B(k^m)$, let

$$a_{k^m}^{i^m} = \frac{1}{\pi^m} \int\limits_{P_m} f(t^m) \prod_{j=1}^{m} \cos\left(k_j t_j - \frac{\pi}{2} i_j\right) dt$$

where $i^m = (i_1, \dots, i_j, \dots, i_m) \in B(k^m)$, be the Fourier coefficients of the function $f(x^m)$, and let

$$U^+_{n^m}(\Lambda; f; x^m) = \frac{1}{\pi^m} \int\limits_{P_m} f(x^m + t^m)\, \Lambda^+_{n^m}(\lambda, t^m)\, dt,$$

$$U_{n^{N-m}}(\Lambda; f; x^{N-m}) = \frac{1}{\pi^{N-m}} \int\limits_{P_{N-m}} f(x^{N-m} + t^{N-m})\, \Lambda_{n^{N-m}}(\lambda, t^{N-m})\, dt,$$

$$U^+_{n^m; n^{N-m}}(\Lambda; f; x) = \frac{1}{\pi^N} \int\limits_{P_N} f(x+t)\, \Lambda^+_{n^m}(\lambda, t^m)\, \Lambda_{n^{N-m}}(\lambda, t^{N-m})\, dt,$$

$$U^+_{n_1; \dots; n_m; n^{N-m}}(\Lambda; f; x) = \frac{1}{\pi^N} \int\limits_{P_N} f(x+t) \prod_{i=1}^{m} \Lambda^+_{n_i}(\lambda, \mu, t_i)\, \Lambda_{n^{N-m}}(\lambda, t^{N-m})\, dt,$$

$$U^+_{n_i}(\lambda; \mu; f; x_i) = \frac{1}{\pi} \int\limits_{0}^{2\pi} f(x_i + t_i)\, \Lambda^+_{n_i}(\lambda, \mu, t_i)\, dt_i$$

be the linear operators with the kernels

$$\Lambda^+_{n^m}(\lambda; t^m) = \sum_{k^m \in E^m} \frac{1}{2^{s(k^m)}} \left(\sum_{l^m \in B(k^m)} \lambda_{k^m}^{l^m} \prod_{i=1}^{m} (-1)^{l_i} \cos\left(k_i t_i - \frac{\pi}{2} l_i\right) \right) \geq 0,$$

$$\Lambda_{n^{N-m}}(\lambda; t^{N-m}) = \sum_{k^{N-m} \in E^{N-m}} \frac{1}{2^{s(k^{N-m})}} \left(\sum_{l^{N-m} \in B(k^{N-m})} \lambda_{k^{N-m}}^{l^{N-m}} \prod_{i=m+1}^{N} (-1)^{l_i} \cos\left(k_i t_i - \frac{\pi}{2} l_i\right) \right),$$

$$\Lambda^+_{n_i}(\lambda; \mu; t_i) = \frac{1}{2} + \sum_{k=1}^{n_i - 1} \left(\lambda_k^{(n_i)} \cos k t_i - \mu_k^{(n_i)} \sin k t_i \right) \geq 0.$$

The operator $U^{+}_{n^m}(\Lambda; f; x^m)$ can be represented in the form

$$U^{+}_{n^m}(\Lambda; f; x^m) = \sum_{k^m \in E^m} \frac{1}{2^{s(k^m)}} \left(\sum_{l^m \in B(k^m)} \lambda^{l^m}_{k^m} \left(\sum_{i^m \in B(k^m)} a^{i^m}_{k^m} \prod_{j=1}^{m} \cos\left(k_j t_j - \frac{\pi}{2}(i_j + l_j) \right) \right) \right).$$

If $f(x) = f(x_i)$ is a function of the single variable x_i, then

$$U^{+}_{n^m}(\Lambda; f; x_i) = \frac{1}{2^m} a_0 + \frac{1}{2^{m-1}} \sum_{k_i=1}^{n_i-1} \left(\lambda_{k_i} \left(a_{k_i} \cos k_i x_i + b_{k_i} \sin k_i x_i \right) + \mu_{k_i} \left(a_{k_i} \sin k_i x_i - b_{k_i} \cos k_i x_i \right) \right),$$

where

$$a_{k_i} = a^{0,\ldots,0,0,0,\ldots,0}_{0,\ldots,0,k_i,0,\ldots,0}, \qquad b_{k_i} = a^{0,\ldots,0,1,0,\ldots,0}_{0,\ldots,0,k_i,0,\ldots,0},$$
$$\lambda_{k_i} = \lambda^{0,\ldots,0,0,0,\ldots,0}_{0,\ldots,0,k_i,0,\ldots,0}, \qquad \mu_{k_i} = \lambda^{0,\ldots,0,1,0,\ldots,0}_{0,\ldots,0,k_i,0,\ldots,0}. \tag{1}$$

Denote

$$U^{+}_{n_i}(\Lambda; f; x_i) = 2^{m-1} U^{+}_{n^m}(\Lambda; f; x_i) = \frac{1}{\pi^m} \int_{P_m} f(x_i + t_i) \Lambda^{+}_{n^m}(\lambda; t^m)\, dt = \frac{1}{\pi} \int_0^{2\pi} f(x_i + t_i) \Lambda^{+}_{n_i}\left(\bar{\lambda}; t_i\right) dt_i,$$

where

$$\Lambda^{+}_{n_i}\left(\bar{\lambda}; t_i\right) = \frac{1}{2} + \sum_{k_i=1}^{n_i-1} \left(\lambda_{k_i} \cos k_i t_i - \mu_{k_i} \sin k_i t_i \right) \geq 0$$

and λ_{k_i} and μ_{k_i} are defined by (1).

Let A be a linear operator that maps a set $M \subset X^N$ $(X^N = C^N,\ L^N_\infty,\ L^N_p)$ onto X^N, let the set M be translation-invariant, i.e., the inclusion $f(x) \in M$ implies that $f(x+t) \in M$, and let

$$G(M, A)_{X^N} = \sup_{f \in M} \| f - A(f) \|_{X^N}$$

be the approximation of the set M by the operator A in the space X^N.

By M_{X^m}, $M_{X^{N-m}}$, and M^{x_i} we denote the subsets of functions from the set M obtained from M by fixing the variables $(x_{m+1}, \ldots, x_N)$, $(x_1, \ldots, x_m)$, and $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, respectively.

In what follows, we prove that if the set M is translation-invariant, then

$$G\left(M, U^{+}_{n^m; n^{N-m}}\right)_{X^N} \leq G\left(M_{X^m}, U^{+}_{n^m}\right)_{X^m} + G\left(M_{X^{N-m}}, U_{n^{N-m}}\right)_{X^{N-m}}. \tag{2}$$

We also consider the cases where inequality (2) turns into the equality.

If we fix the variables $\left(x_1^0,\ldots,x_m^0\right) = x_0^m$ or $\left(x_{m+1}^0,\ldots,x_N^0\right) = x_0^{N-m}$, then the definition of norms yields

$$\|f\|_{C^N} = \sup_{x_0^m}\left\|f\left(x_0^m, x^{N-m}\right)\right\|_{C^{N-m}} = \sup_{x_0^{N-m}}\left\|f\left(x^m, x_0^{N-m}\right)\right\|_{C^m}, \tag{3}$$

$$\|f\|_{L_\infty^N} = \sup_{x_0^m}\operatorname{vrai}\left\|f\left(x_0^m, x^{N-m}\right)\right\|_{L_\infty^{N-m}} = \sup_{x_0^{N-m}}\operatorname{vrai}\left\|f\left(x^m, x_0^{N-m}\right)\right\|_{L_\infty^m}, \tag{4}$$

$$\|f\|_{L_P^N} \le \sup_{x_0^m}\left\|f\left(x_0^m, x^{N-m}\right)\right\|_{L_P^{N-m}}, \tag{5}$$

$$\|f\|_{L_P^N} \le \sup_{x_0^{N-m}}\left\|f\left(x^m, x_0^{N-m}\right)\right\|_{L_P^m}. \tag{6}$$

Lemma 1. *If a set $M \subset X^N$ is translation-invariant, then inequality (2) is true.*

Proof. We introduce the auxiliary linear operators

$$U^+_{n^m;\infty}(\Lambda; f; x) = \frac{1}{\pi^m}\int_{P_m} f(x^m + t^m, x^{N-m})\Lambda^+_{n^m}(\lambda, t^m)\,dt,$$

$$U_{n^{N-m};\infty}(\Lambda; f; x) = \frac{1}{\pi^{N-m}}\int_{P_{N-m}} f(x^m, x^{N-m} + t^{N-m})\Lambda_{n^{N-m}}(\lambda, t^{N-m})\,dt.$$

Using the Fubini theorem, the nonnegativity of the kernel $\Lambda^+_{n^m}(\lambda; t^m)$, and the definition of the operators $U^+_{n^m;\infty}(\Lambda; f; x)$ and $U_{n^{N-m};\infty}(\Lambda; f; x)$, we get

$$\begin{aligned}
&\left\|f(x) - U^+_{n^m;\,n^{N-m}}(\Lambda; f; x)\right\|_{X^N} \\
&\quad\le \left\|\frac{1}{\pi^m}\int_{P_m}\Big(f(x) - f(x^m + t^m; x^{N-m})\Big)\Lambda^+_{n^m}(\lambda, t^m)\,dt^m\right\|_{X^N} + \left\|\frac{1}{\pi^m}\int_{P_m}\Lambda^+_{n^m}(\lambda, t^m)\right. \\
&\qquad\times\left.\left(\frac{1}{\pi^{N-m}}\int_{P_{N-m}}\Big(f(x^m + t^m, x^{N-m}) - f(x+t)\Big)\Lambda_{n^{N-m}}(\lambda, t^{N-m})\,dt^{N-m}\right)dt^m\right\|_{X^N} \\
&\quad= \left\|f(x) - U^+_{n^m;\,\infty}(\Lambda; f; x)\right\|_{X^N} \\
&\qquad+ \left\|\frac{1}{\pi^m}\int_{P_m}\Lambda^+_{n^m}(\lambda, t^m)\Big(f(x^m + t^m, x^{N-m}) - U_{n^{N-m};\,\infty}\big(\Lambda; f; (x^m + t^m, x^{N-m})\big)\Big)dt^m\right\|_{X^N}.
\end{aligned} \tag{7}$$

Using inequality (7) and the Minkowski generalized inequality (see, e.g., [2, p. 22], we obtain

$$G\Big(M, U^{+}_{n^m;\,n^{N-m}}\Big) \le \sup_{f\in M}\Bigg(\Big\| f(x) - U^{+}_{n^m;\,\infty}(\Lambda; f; x)\Big\|_{X^N}$$

$$+ \frac{1}{\pi^m}\int\limits_{P_m} \Lambda^{+}_{n^m}(\lambda, t^m)\Big\| f(x^m + t^m; x^{N-m}) - U_{n^{N-m};\,\infty}\Big(\Lambda; f; \big(x^m + t^m, x^{N-m}\big)\Big)\Big\|_{X^N} dt^m\Bigg). \quad (8)$$

Inequality (8) and relations (3)–(6) yield

$$G\Big(M, U^{+}_{n^m;\,n^{N-m}}\Big) \le \sup_{f\in M}\sup_{x_0^{N-m}}\Big\| f\big(x^m, x_0^{N-m}\big) - U^{+}_{n^m;\,\infty}\Big(\Lambda; f; \big(x^m, x_0^{N-m}\big)\Big)\Big\|_{X^m}$$

$$+ \frac{1}{\pi^m}\int\limits_{P_m} \Lambda^{+}_{n^m}(\lambda, t^m)$$

$$\times \sup_{f\in M}\sup_{x_0^m + t_0^m}\Big\| f\big(x_0^m + t_0^m, x^{N-m}\big) - U_{n^{N-m};\,\infty}\Big(\Lambda; f; \big(x_0^m + t_0^m, x^{N-m}\big)\Big)\Big\|_{X^{N-m}} dt^m.$$

For fixed x_0^{N-m} and $x_0^m + t_0^m$, $f(x)$ belongs to the sets M_{X^m} and $M_{X^{N-m}}$, respectively. Therefore, using the definition of the operators $U^{+}_{n^m;\,\infty}(\Lambda; f; x)$, $U_{n^{N-m};\,\infty}(\Lambda; f; x)$, $U^{+}_{n^m}(\Lambda; f; x^m)$, and $U_{n^{N-m}}(\Lambda; f; x^{N-m})$, we get

$$G\Big(M, U^{+}_{n^m;\,n^{N-m}}\Big)_{X^N} \le \sup_{f\in M_{X^m}}\Big\| f(x) - U^{+}_{n^m}(\Lambda; f; x^m)\Big\|_{X^m}$$

$$+ \frac{1}{\pi^m}\int\limits_{P_m} \Lambda^{+}_{n^m}(\lambda, t^m) \sup_{f\in M_X^{N-m}}\Big\| f(x) - U_{n^{N-m}}(\Lambda; f; x^{N-m})\Big\|_{X^{N-m}} dt^m$$

$$= G\Big(M_{X^m}, U^{+}_{n^m}\Big)_{X^m} + G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}}.$$

Lemma 1 is proved.

Corollary 1. *If a set* $M \subset X^N$ *is translation-invariant, then*

$$G\Big(M, U^{+}_{n_1,\ldots,n_m;\,n^{N-m}}\Big)_{X^N} \le \sum_{i=1}^{m} G\Big(M^{x_i}, U^{+}_{n_i}(\lambda, \mu)\Big)_X + G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}}.$$

The following theorem is true:

Theorem 1. *Suppose that a set* $M \in X^N$ $(X^N = C^N,\ L_\infty^N)$ *is translation-invariant and centrally symmetric, i.e., the inclusion* $f(x) \in M$ *implies that* $-f(x) \in M$. *If the inclusions* $f(x^m) \in M_{X^m} \subset M$ *and* $g(x^{N-m}) \in M_{X^{N-m}} \subset M$ *imply that* $\big(f(x^m) + g(x^{N-m})\big) \in M$, *then*

$$G\Big(M, U^+_{n^m;\,n^{N-m}}\Big)_{X^N} = G\Big(M_{X^m}, U^+_{n^m}\Big)_{X^n} + G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}}. \tag{9}$$

Proof. Assume that $G\Big(M_{X^m}, U^+_{n^m}\Big)_{X^m} = \infty$ or $G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}} = \infty$. Since $M_{X^m} \subset M$ and $M_{X^{N-m}} \subset M$, we get

$$G\Big(M, U^+_{n^m;\,n^{N-m}}\Big)_{X^N} \ge G\Big(M_{X^m}, U^+_{n^m;\,n^{N-m}}\Big)_{X^N} = G\Big(M_{X^m}, U^+_{n^m}\Big)_{X^m} = \infty$$

or

$$G\Big(M, U^+_{n^m;\,n^{N-m}}\Big)_{X^N} \ge G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}}.$$

Lemma 1 yields relation (9).

Let $G\Big(M_{X^m}, U^+_{n^m}\Big)_{X^m} < \infty$ and $G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}} < \infty$. Then, by the definition of least upper bound, there exist functions $\varphi_k(x^m) \in M_{X^m}$ and $\phi(x^{N-m}) \in M_{X^{N-m}}$ such that

$$G\Big(M_{X^m}, U^+_{n^m}\Big)_{X^m} = \big\|\varphi_k(x^m) - U^+_{n^m}(\Lambda; \varphi_k; x^m)\big\|_{X^m}$$

and

$$G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}} = \big\|\phi_k(x^{N-m}) - U_{n^{N-m}}(\Lambda; \phi_k; x^{N-m})\big\|_{X^{N-m}},$$

or there exist sequences of functions $\big\{\varphi_k(x^m)\big\} \in M_{X^m}$ and $\big\{\phi_k(x^{N-m})\big\} \in M_{X^{N-m}}$, $k = 1, 2, 3, \ldots$, for which

$$G\Big(M_{X^m}, U^+_{n^m}\Big)_{X^m} = \lim_{k\to\infty}\big\|\varphi_k(x^m) - U^+_{n^m}(\Lambda; \varphi_k; x^m)\big\|_{X^m} \tag{10}$$

and

$$G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}} = \lim_{k\to\infty}\big\|\phi_k(x^m) - U_{n^{N-m}}(\Lambda; \phi_k; x^{N-m})\big\|_{X^{N-m}}. \tag{11}$$

Setting $f_k(x) = \big(\varphi_k(x^m) + \phi_k(x^{N-m})\big) \in M$ and taking into account that the set M is centrally symmetric and either $X^N = C^N$ or $X^N = L_\infty^N$, we obtain

$$G\Big(M, U^{+}_{n^m;\,n^{N-m}}\Big)_{X^N} \;\geq\; \lim_{k\to\infty}\Big\| f_k(x) - U^{+}_{n^m;\,n^{N-m}}(\Lambda; f; x)\Big\|_{X^N}$$

$$= \lim_{k\to\infty}\Big\| \varphi_k(x^m) - U^{+}_{n^m}(\Lambda; \varphi_k; x^m)\Big\|_{X^m}$$

$$+ \lim_{k\to\infty}\Big\| \phi_k(x^{N-m}) - U_{n^{N-m}}(\Lambda; \phi_k; x^{N-m})\Big\|_{X^{N-m}},$$

which, together with (10), (11), and (2), proves relation (9).

Note that there exists a set $M \subset C^N$ that is translation-invariant, centrally symmetric, and such that the inclusions $f(x^m) \in M_{X^m}$ and $g(x^{N-m}) \in M_{X^{N-m}}$ do not imply that $\big(f(x^m) + g(x^{N-m})\big) \in M.$

Let, e.g., $M = H_{\omega(t,z)} \subset C^2$ be the set of functions $f(x, y)$ that are continuous, 2π-periodic in each variable, and such that $\omega(f; t; z) \leq \omega(t, z)$, where

$$\omega(f; t; z) \;\leq\; \sup_{|h|\leq t,\, |\partial|\leq z} \| f(x+h, y+\partial) - f(x, y)\|_{C^2}$$

and $\omega(t, z)$ is a function of the type of a modulus of continuity. It is known that (see, e.g., [3, p. 124]) that

$$\max\{\omega(t, 0); \omega(0, z)\} \;\leq\; \omega(t, z) \;\leq\; \omega(t, 0) + \omega(0, z).$$

If $\omega(t, z) \neq \omega(t, 0) + \omega(0, z)$, then $f(t) \in \omega(t, 0) \in H_{\omega(t,0)} = M^x$ and $g(z) = \omega(0, z) \in H_{\omega(0,z)} = M^y$, but $(f(t) + g(z)) \notin M.$

Corollary 2. *If a set M is centrally symmetric, translation-invariant, and such that the inclusions $f(x_i) \in M^{x_i} \subset M$, $i = \overline{1, m}$, and $g(x^{N-m}) \in M_{X^{N-m}} \subset M$ imply that $\left(\sum_{i=1}^{m} f(x_i) + g(x^{N-m})\right) \in M$, then*

$$G\Big(M, U^{+}_{n_1,\ldots,n_m;\,n^{N-m}}\Big)_{X^N} \;=\; \sum_{i=1}^{m} G\Big(M^{x_i}, U_{n_i}(\lambda, \mu)\Big)_{x_i} \;+\; G\Big(M_{X^{N-m}}, U_{n^{N-m}}\Big)_{X^{N-m}},$$

where $X^N = C^N$ or $X^N = L^N_\infty$.

Corollary 2 is obtained from Corollary 1 by analogy with the proof of Theorem 1.

Let $H^N_{\omega^{(1)}}$ denote the class of functions $f(x) \in C^N$ that satisfy the condition

$$|f(x) - f(x')| \;\leq\; \sum_{i=1}^{N} \omega_i^{(1)}(|x_i - x_i'|)$$

and let $H^N_{\omega^{(2)}}$ be the class of functions $f(x) \in C^N$ for which

$$|f(x+h) - 2f(x) + f(x-h)| \leq \sum_{i=1}^{N} \omega_i^{(2)}(|h_i|).$$

Here, $\omega_i^{(1)}(t_i)$ and $\omega_i^{(2)}(t_i)$ are arbitrary fixed functions of the type of moduli of continuity of the first order and the second order, respectively. Since the classes $H^N_{\omega^{(i)}}$, $i = 1, 2$, satisfy the conditions of Corollary 2, we can obtain, e.g., Theorem 1 in [1] as a consequence of Corollary 2.

Theorem 2. *If a set N satisfies the conditions of Corollary 2 and, in addition, the inclusion $f(x) \in M$ implies that $(f(x) + C) \in M$, where C is an arbitrary constant, then*

$$G\left(M, U^+_{n^m; n^{N-m}}\right)_{X^N} = \sum_{i=1}^{m} G\left(M^{x_i}, U^+_{n_i}(\Lambda)\right)_{X^1} + G\left(M_{X^{N-m}}, U_{n^{N-m}}\right)_{X^{N-m}}. \tag{12}$$

Since the set $M_{X^m} \in X^N$ is translation-invariant, $X^N = C^N$ or $X^N = L^N_\infty$, and $(f(x^m) + C) \in M_{X^m}$ for every function $f(x^m) \in M_{X^m}$, we get

$$G\left(M_{X^m}, U^+_{n^m}\right)_{X^m} = \sup_{f \in M^0_{X^m}} \left| \frac{1}{\pi^m} \int_{P_m} f(t^m) \Lambda^+_{n^m}\left(\lambda, t^m\right) dt \right|, \tag{13}$$

where $M^0_{X^m}$ is the subset of functions of the set M_{X^m} for which $f(0) = f(0, \ldots, 0) = 0$. By virtue of the fact that the sets M^{x_i} satisfy the same conditions as the set M_{X^m}, taking into account the definition of the operators $U^+_{n_i}(\Lambda; f; x_i)$ we get

$$G\left(M^{x_i}, U^+_{n_i}(\Lambda)\right)_{X^1} = \sup_{f \in M_0^{x_i}} \left| \frac{1}{\pi} \int_0^{2\pi} f(t_i) \Lambda^+_{n_i}\left(\bar{\lambda}, t_i\right) dt_i \right|, \tag{14}$$

where $M_0^{x_i}$ is the subset of functions of the set M^{x_i} for which $f(0) = 0$.

Using the Fubini theorem, we obtain

$$\left| \frac{1}{\pi^m} \int_{P_m} f(t^m) \Lambda^+_{n^m}\left(\lambda, t^m\right) dt \right| = \frac{1}{\pi^{m-1}} \left(\int_{P_{m-1}} \left(\frac{1}{\pi} \int_0^{2\pi} (f(t_1, t_2, \ldots, t_m) - f(0, t_2, \ldots, t_m)) \Lambda^+_{n^m}(\lambda, t^m) dt_1 \right) dt_2 \ldots dt_m \right.$$

$$+ \int_{P_{m-1}} \left(\frac{1}{\pi} \int_0^{2\pi} (f(0, t_2, t_3, \ldots, t_m) - f(0, 0, t_3, \ldots, t_m)) \Lambda^+_{n^m}(\lambda, t^m) dt_2 \right) dt_1 \, dt_3 \ldots dt_m$$

$$\left. + \ldots + \int_{P_{m-1}} \left(\frac{1}{\pi} \int_0^{2\pi} f(0, 0, 0, \ldots, 0, t_m) \Lambda^+_{n^m}(\lambda, t^m) dt_m \right) dt_1 \ldots dt_{m-1} \right). \tag{15}$$

If $f(t_1, \ldots, t_m) \in M^0_{X^m}$, then, for fixed $(t_2, \ldots, t_m), (t_3, \ldots, t_m), \ldots, t_m$, we obtain $(f(t_1, t_2, \ldots, t_m) - f(0, t_2, \ldots, t_m)) \in M_0^{t_1}$, $(f(0, t_2, t_3, \ldots, t_m) - f(0, 0, t_3, \ldots, t_m)) \in M_0^{t_2}, \ldots, f(0, 0, 0, \ldots, t_m) \in M_0^{t_m}$, respectively. Then, taking into account relation (15), the nonnegativity of the kernel $\Lambda^+_{n^m}(\lambda, t^m)$, and the definition of the operators $U^+_{n_i}(\Lambda; f; x_i)$, we obtain

$$\sup_{f \in M^0_{X^m}} \left| \frac{1}{\pi^m} \int_{P_m} f(t^m) \Lambda^+_{n^m}\left(\lambda, t^m\right) dt \right| \le \sum_{i=1}^{m} \sup_{f \in M_0^{t_i}} \left| \frac{1}{\pi^m} \int_{P_m} f(t_i) \Lambda_{n^m}\left(\lambda, t^m\right) dt \right|$$

$$= \sum_{i=1}^{m} \sup_{f \in M_0^{t_i}} \left| \frac{1}{\pi} \int_0^{2\pi} f(t_i) \Lambda^+_{n_i}\left(\bar{\lambda}, t_i\right) dt_i \right|. \tag{16}$$

Relations (13), (16), and (14) and Theorem 1 yield

$$G\left(M, U^+_{n^m; n^{N-m}}\right)_{X^N} \le \sum_{i=1}^{m} G\left(M^{x_i}, U^+_{n_i}(\Lambda)\right)_{X^1} + G\left(M_{X^{N-m}}, U_{n^{N-m}}\right)_{X^{N-m}}.$$

The proof of equality (12) is analogous to that of equality (9).

Theorem 2 is proved.

Remark 1. For classes of functions that satisfy the conditions of Theorem 2, the approximation by a linear positive operator with arbitrary kernel depends on the one-dimensional terms of this kernel. Therefore, according to Corollary 2, for such classes of functions the approximation by a positive operator with arbitrary kernel coincides with the approximation by a positive operator whose kernel is the product of one-dimensional kernels.

By using the statements proved, we can find, e.g., upper bounds or asymptotic equalities for the quantities $G\left(M, U^+_{n^m; n^{N-m}}\right)_{X^N}$, provided that upper bounds or asymptotic equalities, respectively, are known for the quantities $G\left(M_{X^m}, U^+_{n^m}\right)_{X^m}$, $G\left(M_{X^{N-m}}, U_{n^{N-m}}\right)_{X^{N-m}}$, and $G\left(M^{x_i}, U^+_{n_i}(\Lambda)\right)_X$, $i = \overline{1, m}$.

Consider the Fejér operator

$$F_l(f, x_1) = \frac{1}{\pi} \int_0^{2\pi} f(x_1 + t_1) \Lambda^+_l(t_1) dt_1$$

with the kernel

$$\Lambda^+_l(t_1) = \frac{\sin^2(lt_1/2)}{2l \sin(t_1/2)},$$

the Fourier operator

$$S_{n,m}(f, x_2, x_3) = \frac{1}{\pi^2} \iint_{P_2} f(x_2 + t_2, x_3 + t_3) D_n(t_2) D_m(t_3) dt_2\, dt_3$$

with the Dirichlet kernel

$$D_k(t) = \frac{\sin(k+1/2)t}{2\sin(t/2)},$$

and the operator

$$F_l S_{n,m}(f,\ x^3) = \frac{1}{\pi^3}\int_{P_3} f(x^3+t^3)\Lambda^+_{l,n,m}(t^3)\,dt^3$$

with the kernel

$$\Lambda^+_{l,n,m}(t^3) = \Lambda^+_l(t_1)D_n(t_2)D_m(t_3),$$

which is the product of the Fejér kernel and the Dirichlet kernels. Then Corollary 2 yields

$$G\Big(H^3_{\omega^{(1)}},\, F_l\, S_{n,m}\Big)_{C^3} = G\Big(H^1_{\omega^{(1)}},\, F_l\Big)_{C^1} + G\Big(H^2_{\omega^{(1)}},\, S_{n,m}\Big)_{C^2}.$$

Note that additional information about asymptotic equalities for the quantities $G\Big(H^2_{\omega^{(1)}},\, S_{n,m}\Big)_{C^2}$ and $G\Big(H^1_{\omega^{(1)}},\, F_l\Big)_{C^1}$ and the corresponding bibliography can be found in [4].

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