Ukrainian Mathematical Journal, Vol. 56, No. 9, 2004

APPROXIMATION OF FUNCTIONS DEFINED ON THE REAL AXIS BY OPERATORS GENERATED BY λ-METHODS OF SUMMATION OF THEIR FOURIER INTEGRALS

Yu. I. Kharkevych and T. V. Zhyhallo

UDC 517.5

We obtain asymptotic equalities for upper bounds of the deviations of operators generated by λ-methods (defined by a collection $\Lambda = \{\lambda_\sigma(\cdot)\}$ of functions continuous on $[0;\infty)$ and depending on a real parameter σ) on classes of (ψ,β)-differentiable functions defined on the real axis.

1. Auxiliary Assertions and Statement of the Problem

For many years, Stepanets and his followers have investigated the approximation properties of the classes $L^\psi_\beta\mathfrak{N}$ and $\hat{L}^\psi_\beta\mathfrak{N}$ defined by the property that the generalized (ψ,β)-derivatives of their elements belong to a certain set $\mathfrak{N}$. For numerous results concerning these problems, see [1–9].

According to [3] (Chap. IX), the classes $\hat{L}^\psi_\beta\mathfrak{N}$ are defined as follows: Let $L_p, p \geq 1$, be the set of 2π-periodic functions $\varphi(\cdot)$ with finite norm $\|\varphi\|_p$, where

$$\|\varphi\|_p = \left(\int_0^{2\pi} |\varphi(t)|^p dt\right)^{1/p} \qquad \text{for} \qquad p \in [1;\infty)$$

and $\|\varphi\|_\infty = \|\varphi\|_M = \operatorname{ess\,sup}|\varphi(t)|$, so that $L_\infty = M$.

The spaces $\hat{L}_p$, $p \geq 1$, are introduced as the sets of (not necessarily periodic) functions $\varphi(\cdot)$ defined on the entire real axis R and having a finite norm $\|\varphi\|_{\hat{p}}$, where

$$\|\varphi\|_{\hat{p}} = \sup_{a\in R}\left(\int_a^{a+2\pi} |\varphi(t)|^p dt\right)^{1/p} \qquad \text{for} \qquad p \in [1,\infty)$$

and $\|\varphi\|_{\hat{\infty}} = \operatorname{ess\,sup}|\varphi(t)|$.

It is obvious that, for all $p \geq 1$, the inclusion $L_p \subset \hat{L}_p$ is always true.

Let $\mathfrak{M}$ denote the set of functions $\psi(v)$ convex downward for all $v \geq 1$ and such that

$$\lim_{v\to\infty} \psi(v) = 0.$$

We extend every function $\psi \in \mathfrak{M}$ to the segment $[0,1)$ so that the function obtained (denoted, as before, by $\psi(\cdot)$) is continuous for all $v \geq 0$, $\psi(0) = 0$, and its derivative $\psi'(v) = \psi'(v+0)$ has a small variation on the segment $[0,\infty)$. Denote the set of these functions by $\mathfrak{A}$. The subset of functions $\psi \in \mathfrak{A}$ for which

Volyn University, Luts'k. Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 56, No. 9, pp. 1267–1280, September, 2004. Original article submitted November 11, 2003.

$$\int_1^\infty \frac{\psi(t)}{t}\,dt < \infty$$

is denoted by F.

We set

$$\hat{\psi}(t) = \hat{\psi}_\beta(t) = \frac{1}{\pi}\int_0^\infty \psi(v)\cos\left(vt + \frac{\beta\pi}{2}\right)dv,$$

where $\psi \in F$ and β is a certain fixed number.

If $\psi \in F$, then, as shown in [4], for any $\beta \in R$ the transformation $\hat{\psi}(t)$ is summable on the entire axis:

$$\int_{-\infty}^\infty |\hat{\psi}(t)|\,dt < \infty.$$

Let $\hat{L}^\psi_\beta$ denote the set of functions $f(x) \in \hat{L}_1$ that, for almost all $x \in R$, can be represented in the form

$$f(x) = A_0 + \int_{-\infty}^\infty \varphi(x+t)\,\hat{\psi}(t)\,dt, \tag{1}$$

where A_0 is a certain constant, $\varphi(\cdot) \in \hat{L}_1$, and the integral is understood as the limit of integrals over symmetrically expanding intervals.

If $f(\cdot) \in \hat{L}^\psi_\beta$ and, in addition, $\psi \in \mathfrak{N}$, where $\mathfrak{N}$ is a certain subset of continuous functions from $\hat{L}_1$, then we assume that $f(\cdot) \in \hat{L}^\psi_\beta \mathfrak{N}$. The subsets of continuous functions from $\hat{L}^\psi_\beta$ $\left(\hat{L}^\psi_\beta \mathfrak{N}\right)$ are denoted by $\hat{C}^\psi_\beta$ $\left(\hat{C}^\psi_\beta \mathfrak{N}\right)$. If $\mathfrak{N}$ coincides with the set of functions $\varphi(\cdot)$ satisfying the condition $\operatorname{ess\,sup}|\varphi(\cdot)| \le 1$, then the class $\hat{C}^\psi_\beta \mathfrak{N}$ is denoted by $\hat{C}^\psi_{\beta,\infty}$. If $f \in \hat{L}^\psi_\beta$ and $\left\| f^\psi_\beta \right\|_{\hat{1}} \le 1$, then we say that $f \in \hat{L}^\psi_{\beta,1}$.

In [3] (Chap. IX), it was shown that if $\varphi(\cdot)$ is a 2π-periodic summable function, then the sets $\hat{L}^\psi_\beta \mathfrak{N}$, $\hat{L}^\psi_{\beta,1}$, and $\hat{C}^\psi_{\beta,\infty}$ transform into the classes $L^\psi_\beta \mathfrak{N}$, $L^\psi_{\beta,1}$, and $C^\psi_{\beta,\infty}$, respectively. In the periodic case where relation (1) holds, we have $\varphi(\cdot) = f^\psi_\beta(\cdot)$ almost everywhere. In this connection, any function equivalent to the function $\varphi(\cdot)$ in relation (1) is called, as in the periodic case [see, e.g., [1] (Chap. I) and [4] (Chap. III)], the (ψ, β)-derivative of $f(\cdot)$ and is denoted by $f^\psi_\beta(\cdot)$.

As mentioned above, the classes $\hat{L}^\psi_\beta \mathfrak{N}$ were introduced by Stepanets. He also considered the problem of the approximation of functions from the classes $\hat{L}^\psi_\beta \mathfrak{N}$ by using the so-called Fourier operators, which, in the periodic case, are Fourier sums of order $[\sigma]$; in the general case, they are entire functions of exponential type $\le \sigma$ (see [4, 5]). In these works, Stepanets obtained a representation on the classes $\hat{L}^\psi_\beta \mathfrak{N}$ for the deviations of the operators $U_\sigma(f, x, \lambda)$, which are integral analogs of the polynomial operators generated by triangular λ-methods of summation of Fourier series. These results were applied in [6–9] to the problem of approximation of functions from the classes $\hat{L}^\psi_\beta \mathfrak{N}$ by the operators of Zygmund, Steklov, de la Vallée-Poussin, etc.

The aim of the present paper is to study the deviations (on the classes $\hat{L}^{\psi}_{\beta,1}$ and $\hat{C}^{\psi}_{\beta,\infty}$) of the operators $U_\sigma(f, x, \lambda)$ generated by λ-methods (defined by a collection $\Lambda = \{\lambda_\sigma(\cdot)\}$ of functions continuous on $[0, \infty)$ and depending on a real parameter σ) of summation of Fourier integrals. In the periodic case, for $\psi(v) = v^{-r}$, $r > 0$, the most complete results in this direction were obtained in [10]; for functions ψ decreasing to zero, the most complete results were obtained in [11].

Let $\Lambda = \left\{\lambda_\sigma\left(\frac{v}{\sigma}\right)\right\}$ be a collection of functions continuous for all $v \geq 0$ and depending on a real parameter σ. We associate every function $f \in \hat{L}^{\psi}_{\beta}$ with the expression

$$U_\sigma(\Lambda) = U_\sigma(f, x, \Lambda) = A_0 + \int_{-\infty}^{\infty} f^{\psi}_{\beta}(x+t)\,\frac{1}{\pi}\int_0^{\infty} \psi(v)\lambda_\sigma\left(\frac{v}{\sigma}\right)\cos\left(vt + \frac{\beta\pi}{2}\right) dv\,dt. \tag{2}$$

Further, we assume that the functions $\psi(v)$ and $\lambda_\sigma\left(\frac{v}{\sigma}\right)$ are such that the transformations

$$\widehat{\psi\lambda_\sigma} = \frac{1}{\pi}\int_0^{\infty} \psi(v)\lambda_\sigma\left(\frac{v}{\sigma}\right)\cos\left(vt + \frac{\beta\pi}{2}\right) dv$$

are summable on the entire number axis.

Then, using relations (1) and (2), for every function $f(\cdot) \in \hat{C}^{\psi}_{\beta}$ we obtain

$$f(x) - U_\sigma(f, x, \Lambda) = \int_{-\infty}^{\infty} f^{\psi}_{\beta}(x+t)\,\frac{1}{\pi}\int_0^{\infty} r_\sigma\left(\frac{v}{\sigma}\right)\cos\left(vt + \frac{\beta\pi}{2}\right) dv\,dt, \tag{3}$$

where, for $v \geq 1$,

$$r_\sigma\left(\frac{v}{\sigma}\right) = \left(1 - \lambda_\sigma\left(\frac{v}{\sigma}\right)\right)\psi(v), \tag{4}$$

and, on the segment $0 \leq v \leq 1$, the function $r_\sigma\left(\frac{v}{\sigma}\right)$ is arbitrarily defined so that it is continuous for all $v \geq 0$, equal to zero at the origin, and such that its Fourier transform

$$\hat{r}_\sigma(t) = \frac{1}{\pi}\int_0^{\infty} r_\sigma\left(\frac{v}{\sigma}\right)\cos\left(vt + \frac{\beta\pi}{2}\right) dv$$

is summable on the entire number axis.

In the present paper, we investigate the quantities

$$\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}, U_\sigma(\Lambda)\right)_C = \sup_{f \in \hat{C}^{\psi}_{\beta,\infty}} \| f(x) - U_\sigma(f, x, \Lambda)\|_C, \tag{5}$$

$$\mathscr{E}\Big(\hat{L}^{\psi}_{\beta,1}, U_{\sigma}(\Lambda)\Big)_{\hat{1}} = \sup_{f \in \hat{L}^{\psi}_{\beta,1}} \| f(x) - U_{\sigma}(f, x, \Lambda)\|_{\hat{1}}, \tag{6}$$

where $U_{\sigma}(f, x, \Lambda)$ are the operators defined by (2).

First, we present several auxiliary definitions and statements necessary for what follows.

Definition 1 [10]. *Let a function* $\tau(v)$ *be defined on* $[0, \infty)$, *absolutely continuous, and such that* $\tau(\infty) = 0$. *We say that* $\tau(v) \in \mathscr{E}_a$ *if the derivative* $\tau'(v)$ *can be defined at the points where it does not exist so that, for a certain* $a \geq 0$, *the following integrals exist:*

$$\int_0^{a/2} v\left|d\tau'(v)\right|, \qquad \int_{a/2}^{\infty} |v - a|\left|d\tau'(v)\right|.$$

Let K and K_i denote constants that are, generally speaking, different in different relations.

Lemma 1′ [10]. *If* $\tau(v) \in \mathscr{E}_a$, *then*

$$|\tau(v)| \leq |\tau(0)| + |\tau(a)| + \int_0^{a/2} v\left|d\tau'(v)\right| + \int_{a/2}^{\infty} |v - a|\left|d\tau'(v)\right| =: H(\tau). \tag{7}$$

Theorem 1′ [10]. *Suppose that* $\tau(v) \in \mathscr{E}_a$ *and* $\sin\frac{\beta\pi}{2}\tau(0) = 0$. *In order that the integral*

$$A(\tau) = \frac{1}{\pi}\int_0^{\infty}\left|\int_0^{\infty}\tau(v)\cos\Big(vt + \frac{\beta\pi}{2}\Big)\,dv\right| dt \tag{8}$$

be convergent, it is necessary and sufficient that the integrals

$$\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\infty}\frac{|\tau(v)|}{v}\,dv, \qquad \int_0^{a}\frac{|\tau(a - v) - \tau(a + v)|}{v}\,dv$$

be convergent. In this case, the following estimate is true:

$$\left|A(\tau) - \frac{4}{\pi^2}\int_0^{\infty}\xi\Big(\sin\frac{\beta\pi}{2}\tau(v),\ j_v[\tau(a - v) - \tau(a + v)]\Big)\frac{dv}{v}\right| \leq KH(\tau), \tag{9}$$

where $\xi(A, B)$ *is the function defined as follows [12]:*

$$\xi(A, B) = \begin{cases} \dfrac{\pi}{2}|A|, & |B| \leq |A|, \\[2mm] |A|\arcsin\left|\dfrac{A}{B}\right| + \sqrt{B^2 - A^2}, & |B| > |A|, \end{cases} \tag{10}$$

$$j_v = \begin{cases} 1, & 0 < v < a, \\ 0, & v \geq a. \end{cases} \tag{11}$$

Let $\psi \in \mathfrak{M}$. Following [2, pp. 159, 160], we set

$$\eta(t) := \psi^{-1}\left(\frac{\psi(t)}{2}\right), \qquad \mu(t) := \frac{t}{\eta(t) - t},$$

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M}: 0 < \mu(\psi, t) \leq K \quad \forall\, t \geq 1\},$$

$$\mathfrak{M}_c = \{\psi \in \mathfrak{M}: 0 < K_1 < \mu(\psi, t) \leq K_2 \quad \forall\, t \geq 1\}.$$

If $\psi \in \mathfrak{A}$ and, moreover, $\psi \in \mathfrak{M}_0$ or $\psi \in \mathfrak{M}_C$ for $t \geq 1$, then, following [4, p. 112], we write $\psi \in \mathfrak{A}_0$ or $\psi \in \mathfrak{A}_C$, respectively.

Theorem 2′ [2, p. 161]. *A function $\psi \in \mathfrak{M}$ belongs to $\mathfrak{M}_0$ if and only if the quantity*

$$\alpha(t) = \frac{\psi(t)}{t|\psi'(t)|}, \qquad \psi'(t) := \psi'(t+0),$$

satisfies the condition

$$\alpha(t) \geq K > 0 \qquad \forall\, t \geq 1.$$

Theorem 3′ [2, p. 175]. *In order that a function $\psi \in \mathfrak{M}$ belong to $\mathfrak{M}_0$, it is necessary and sufficient that there exist a constant K such that, for all $t \geq 1$, the following inequality is true:*

$$\frac{\psi(t)}{\psi(ct)} \leq K,$$

where c is an arbitrary constant that satisfies the condition $c > 1$.

2. Asymptotic Relations for $\mathscr{E}\left(\hat{C}^{\psi}_{\beta,\infty}, U_\sigma(\Lambda)\right)_C$

For convenience, performing a change of variables, we rewrite relations (3) and (4) in the form

$$f(x) - U_\sigma(f, x, \Lambda) = \psi(\sigma) \int_{-\infty}^{\infty} f_\beta^\psi\left(x + \frac{t}{\sigma}\right) \frac{1}{\pi} \int_0^\infty \tau_\sigma(v) \cos\left(vt + \frac{\beta\pi}{2}\right) dv\, dt, \tag{12}$$

$$\tau(v) = \tau_\sigma(v) = (1 - \lambda_\sigma(v)) \frac{\psi(\sigma v)}{\psi(\sigma)}, \qquad v \geq \frac{1}{\sigma}, \tag{13}$$

where, as before, the function $\tau_\sigma(v)$ is arbitrarily defined on the segment $\left[0, \frac{1}{\sigma}\right]$ so that it is continuous for all $v \geq 0$, equal to zero at the origin, and such that its Fourier transform

$$\hat{\tau}(t) = \frac{1}{\pi}\int_0^\infty \tau(v)\cos\left(vt + \frac{\beta\pi}{2}\right)dv \tag{14}$$

is summable on the entire number axis.

Then the following theorem is true:

Theorem 1. *Suppose that the following conditions are satisfied:*

(i) $\psi(v) \in F \cap \mathfrak{A}_0$;

(ii) $\tau(v) \in \mathcal{E}_a$;

(iii) $\sin\frac{\beta\pi}{2}\tau(0) = 0$;

(iv) the following integrals converge:

$$\left|\sin\frac{\beta\pi}{2}\right|\int_0^\infty \frac{|\tau_\sigma(v)|}{v}\,dv, \qquad \int_0^a \frac{|\lambda_\sigma(a-v) - \lambda_\sigma(a+v)|}{v}\,dv. \tag{15}$$

Then the function

$$\tau(v) = \tau_\sigma(v) = \begin{cases} (1 - \lambda_\sigma(v))\dfrac{\psi(1)}{\psi(\sigma)}, & 0 \leq v \leq \dfrac{1}{\sigma}, \\ (1 - \lambda_\sigma(v))\dfrac{\psi(\sigma v)}{\psi(\sigma)}, & v \geq \dfrac{1}{\sigma}, \end{cases} \tag{16}$$

satisfies the following relation:

$$\mathcal{E}\left(\hat{C}^\psi_{\beta,\infty}, U_\sigma(\Lambda)\right)_C$$

$$= \frac{4}{\pi^2}\psi(\sigma)\int_0^\infty \xi\left(\sin\frac{\beta\pi}{2}\tau_\sigma(v),\ j_v[\tau_\sigma(a-v) - \tau_\sigma(a+v)]\right)\frac{dv}{v} + O\big(\psi(\sigma)H(\tau_\sigma)\big), \quad \sigma \to \infty, \tag{17}$$

where $H(\tau_\sigma)$, $\xi(A, B)$, *and* j_v *are defined by (7), (10), and (11), respectively.*

Proof. Using Theorem 1′, we show that the integral $A(\tau_\sigma)$ converges, and, hence, by virtue of Lemma 1 in [8], the following relation holds as $\sigma \to \infty$:

$$\mathcal{E}\left(\hat{C}^\psi_{\beta,\infty}, U_\sigma(\Lambda)\right)_C = \psi(\sigma)A(\tau_\sigma). \tag{18}$$

One of the conditions of Theorem 1′ is the convergence of the integral

$$\int_0^a \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv, \tag{19}$$

whereas one of the conditions of Theorem 1 is the convergence of the integral

$$\int_0^a \frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv. \tag{20}$$

Let us show that if $\psi \in \mathfrak{M}_0$, then

$$\int_0^a \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv = \frac{\psi(\sigma a)}{\psi(\sigma)}\int_0^a \frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv + H(\tau_\sigma)O(1), \tag{21}$$

where $O(1)$ is a quantity uniformly bounded in σ. Therefore, the convergence of integral (20) yields the convergence of integral (19).

Using relation (16), we get

$$\tau_\sigma(a-v) = \begin{cases} (1-\lambda_\sigma(a-v))\,\dfrac{\psi(1)}{\psi(\sigma)}, & a-\dfrac{1}{\sigma}\le v\le a, \\ (1-\lambda_\sigma(a-v))\,\dfrac{\psi(\sigma(a-v))}{\psi(\sigma)}, & v\le a-\dfrac{1}{\sigma}, \end{cases} \tag{22}$$

$$\tau_\sigma(a+v) = \begin{cases} (1-\lambda_\sigma(a+v))\,\dfrac{\psi(1)}{\psi(\sigma)}, & -a\le v\le \dfrac{1}{\sigma}-a, \\ (1-\lambda_\sigma(a+v))\,\dfrac{\psi(\sigma(a+v))}{\psi(\sigma)}, & v\ge \dfrac{1}{\sigma}-a. \end{cases} \tag{23}$$

First, we consider the case $a > \dfrac{1}{\sigma}$ and represent relation (19) as a sum of two integrals:

$$\int_0^a \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv = \int_0^{a-\frac{1}{\sigma}} \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv + \int_{a-\frac{1}{\sigma}}^a \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv. \tag{24}$$

Let us estimate the first term on the right-hand side of (24). To this end, we add and subtract the quantity

$$\frac{\psi(\sigma a)}{\psi(\sigma)}\big(\lambda_\sigma(a-v)-\lambda_\sigma(a+v)\big)$$

under the modulus sign in the integrand. As a result, we obtain

$$\int_0^{a-\frac{1}{\sigma}} \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv$$

$$= \frac{\psi(\sigma a)}{\psi(\sigma)} \int_0^{a-\frac{1}{\sigma}} \frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv$$

$$+ \; O\left(\int_0^{a-\frac{1}{\sigma}} \frac{\left|\tau_\sigma(a-v)-\tau_\sigma(a+v)+\dfrac{\psi(\sigma a)}{\psi(\sigma)}\left(\lambda_\sigma(a-v)-\lambda_\sigma(a+v)\right)\right|}{v}\,dv \right). \qquad (25)$$

Since relations (22) and (23) are true, for $v \in \left[0, a-\dfrac{1}{\sigma}\right]$ we get

$$\lambda_\sigma(a-v) = 1 - \frac{\psi(\sigma)}{\psi(\sigma(a-v))}\,\tau_\sigma(a-v)$$

and

$$\lambda_\sigma(a+v) = 1 - \frac{\psi(\sigma)}{\psi(\sigma(a+v))}\,\tau_\sigma(a+v).$$

Then

$$\int_0^{a-\frac{1}{\sigma}} \frac{\left|\tau_\sigma(a-v)-\tau_\sigma(a+v)+\dfrac{\psi(\sigma a)}{\psi(\sigma)}\left(\lambda_\sigma(a-v)-\lambda_\sigma(a+v)\right)\right|}{v}\,dv$$

$$\le \int_0^{a-\frac{1}{\sigma}} |\tau_\sigma(a-v)| \left|1-\frac{\psi(\sigma a)}{\psi(\sigma(a-v))}\right| \frac{dv}{v} + \int_0^{a-\frac{1}{\sigma}} |\tau_\sigma(a+v)| \left|1-\frac{\psi(\sigma a)}{\psi(\sigma(a+v))}\right| \frac{dv}{v}. \qquad (26)$$

Since $\tau_\sigma(v) \in \mathcal{E}_a$, according to Lemma 1′ we get

$$\int_0^{a-\frac{1}{\sigma}} |\tau_\sigma(a-v)| \left|1-\frac{\psi(\sigma a)}{\psi(\sigma(a-v))}\right| \frac{dv}{v} + \int_0^{a-\frac{1}{\sigma}} |\tau_\sigma(a+v)| \left|1-\frac{\psi(\sigma a)}{\psi(\sigma(a+v))}\right| \frac{dv}{v}$$

$$= H(\tau_\sigma)\, O\left(\int_0^{a-\frac{1}{\sigma}} \frac{|\psi(\sigma(a-v))-\psi(\sigma a)|}{v\psi(\sigma(a-v))}\,dv + \int_0^{a-\frac{1}{\sigma}} \frac{|\psi(\sigma(a+v))-\psi(\sigma a)|}{v\psi(\sigma(a+v))}\,dv \right). \qquad (27)$$

Let us show that, as $\sigma \to \infty$,

$$I_{1,\sigma} := \int_0^{a-\frac{1}{\sigma}} \frac{|\psi(\sigma(a-v)) - \psi(\sigma a)|}{v\psi(\sigma(a-v))} dv = O(1), \tag{28}$$

$$I_{2,\sigma} := \int_0^{a-\frac{1}{\sigma}} \frac{|\psi(\sigma(a+v)) - \psi(\sigma a)|}{v\psi(\sigma(a+v))} dv = O(1), \tag{29}$$

where the quantity $O(1)$ is uniformly bounded in σ. Indeed, the function $\dfrac{1-\psi(\sigma a)/\psi(\sigma(a-v))}{v}$ is bounded for all $v \in \left[\delta, a - \dfrac{1}{\sigma}\right]$, $0 < \delta < a - \dfrac{1}{\sigma}$, and, furthermore, by virtue of Theorem 2′, we have

$$\lim_{v\to 0} \frac{1-\psi(\sigma a)/\psi(\sigma(a-v))}{v} = \frac{\sigma\left|\psi'(\sigma a)\right|}{\psi(\sigma a)} \leq K.$$

Therefore, $I_{1,\sigma} = O(1)$ as $\sigma \to \infty$.

Passing to the estimation of the integral $I_{2,\sigma}$, we note that

$$I_{2,\sigma} < \frac{1}{\psi(2a\sigma - 1)} \int_0^{a-\frac{1}{\sigma}} \frac{\psi(a\sigma) - \psi(\sigma(a+v))}{v} dv.$$

Performing the change of variables $u = \sigma(a+v)$, we get

$$I_{2,\sigma} < \frac{1}{\psi(2a\sigma - 1)} \int_{a\sigma}^{2a\sigma-1} \frac{\psi(a\sigma) - \psi(u)}{u - a\sigma} du < \frac{1}{\psi(2a\sigma - 1)} \int_{a\sigma}^{2a\sigma} \frac{\psi(a\sigma) - \psi(u)}{u - a\sigma} du.$$

Using Lemma 1.5 from [13] and Theorem 3′ for the right-hand side of the last inequality, we obtain

$$I_{2,\sigma} < \frac{K_1\psi(a\sigma)}{\psi(2a\sigma - 1)} \leq \frac{K_2\psi(a\sigma)}{\psi(2a\sigma)} \leq K_3.$$

Thus, equalities (28) and (29) are true.

Combining relations (25)–(29), we get

$$\int_0^{a-\frac{1}{\sigma}} \frac{|\tau_\sigma(a-v) - \tau_\sigma(a+v)|}{v} dv = \frac{\psi(\sigma a)}{\psi(\sigma)} \int_0^{a-\frac{1}{\sigma}} \frac{|\lambda_\sigma(a-v) - \lambda_\sigma(a+v)|}{v} dv + H(\tau_\sigma)O(1). \tag{30}$$

Let us estimate the second term on the right-hand side of (24). It is obvious that

$$\int_{a-\frac{1}{\sigma}}^{a} \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv$$

$$= \frac{\psi(\sigma a)}{\psi(\sigma)} \int_{a-\frac{1}{\sigma}}^{a} \frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv$$

$$+ O\left(\int_{a-\frac{1}{\sigma}}^{a} \frac{\left|\tau_\sigma(a-v)-\tau_\sigma(a+v)+\frac{\psi(\sigma a)}{\psi(\sigma)}(\lambda_\sigma(a-v)-\lambda_\sigma(a+v))\right|}{v}\,dv \right). \tag{31}$$

Using relations (22) and (23), for $v \in \left[a-\frac{1}{\sigma}, a\right]$ we get

$$\lambda_\sigma(a-v) = 1 - \frac{\psi(\sigma)}{\psi(1)}\tau_\sigma(a-v) \tag{32}$$

and

$$\lambda_\sigma(a+v) = 1 - \frac{\psi(\sigma)}{\psi(\sigma(a+v))}\tau_\sigma(a+v).$$

Hence, by virtue of Lemma 1′, we obtain

$$\int_{a-\frac{1}{\sigma}}^{a} \frac{\left|\tau_\sigma(a-v)-\tau_\sigma(a+v)+\frac{\psi(\sigma a)}{\psi(\sigma)}(\lambda_\sigma(a-v)-\lambda_\sigma(a+v))\right|}{v}\,dv$$

$$= \int_{a-\frac{1}{\sigma}}^{a} \frac{\left|\tau_\sigma(a-v)\left(1-\frac{\psi(\sigma a)}{\psi(1)}\right)-\tau_\sigma(a+v)\left(1-\frac{\psi(\sigma a)}{\psi(\sigma(a+v))}\right)\right|}{v}\,dv$$

$$= H(\tau_\sigma)O\left(\int_{a-\frac{1}{\sigma}}^{a} \frac{|\psi(1)-\psi(\sigma a)|}{v\psi(1)}\,dv + \int_{a-\frac{1}{\sigma}}^{a} \frac{|\psi(\sigma(a+v))-\psi(\sigma a)|}{v\psi(\sigma(a+v))}\,dv \right). \tag{33}$$

We estimate the right-hand side of (33) as follows:

$$\int_{a-\frac{1}{\sigma}}^{a} \frac{|\psi(1)-\psi(\sigma a)|}{v\psi(1)}\,dv = \left(1-\frac{\psi(\sigma a)}{\psi(1)}\right)\ln\frac{a}{a-\frac{1}{\sigma}} = O(1). \tag{34}$$

By analogy with the proof of relation (29), we get

$$\int_{a-\frac{1}{\sigma}}^{a} \frac{|\psi(\sigma(a+v))-\psi(\sigma a)|}{v\psi(\sigma(a+v))}\,dv \le \frac{1}{\psi(2a\sigma)}\int_{2a\sigma-1}^{2a\sigma}\frac{|\psi(u)-\psi(\sigma a)|}{u-a\sigma}\,du$$

$$\le \frac{1}{\psi(2a\sigma)}\int_{a\sigma}^{2a\sigma}\frac{|\psi(u)-\psi(\sigma a)|}{u-a\sigma}\,du \le \frac{\psi(a\sigma)}{\psi(2a\sigma)} = O(1). \tag{35}$$

Using relations (31)–(35), we obtain

$$\int_{a-\frac{1}{\sigma}}^{a} \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv = \frac{\psi(\sigma a)}{\psi(\sigma)}\int_{a-\frac{1}{\sigma}}^{a}\frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv + H(\tau_\sigma)O(1). \tag{36}$$

Combining relations (36) and (30), we arrive at equality (21).

By analogy with the proof of relation (21), for $a>\frac{1}{\sigma}$ one can show that equality (21) is also valid for $\frac{1}{2\sigma}<a\le\frac{1}{\sigma}$.

For $0<a\le\frac{1}{2\sigma}$, relation (32) is true and

$$\lambda_\sigma(a+v) = 1 - \frac{\psi(\sigma)}{\psi(1)}\tau_\sigma(a+v).$$

Then

$$\int_0^a \frac{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}{v}\,dv = \frac{\psi(1)}{\psi(\sigma)}\int_0^a\frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv.$$

The convergence of integral (20) yields the convergence of integral (19). Thus, for $a\ge 0$, all conditions of Theorem 1′ are satisfied. Then, substituting relation (9) into equality (18), we get (17).

Theorem 1 is proved.

Note that an analogous theorem was proved in [10] in the periodic case where $\psi(v)=v^{-r}$, $r>0$, and in [11] for the classes $C^{\psi}_{\beta,\infty}$, $\psi\in\mathfrak{M}_C$.

Corollary 1. *Suppose that the conditions of Theorem 1 are satisfied. If*

$$|\tau_\sigma(a-v)-\tau_\sigma(a+v)| \le \left|\sin\frac{\beta\pi}{2}\right||\tau_\sigma(v)|, \qquad v\in[0,a], \quad a>0, \tag{37}$$

then

$$\mathscr{E}_\sigma\left(\hat{C}^\psi_{\beta,\infty}, U_\sigma(\Lambda)\right)_C = \frac{2}{\pi}\psi(\sigma)\left|\sin\frac{\beta\pi}{2}\right|\int_0^\infty \frac{|\tau_\sigma(v)|}{v}\,dv + O\left(\psi(\sigma)\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\frac{2}{\sigma\pi}} \frac{|\tau_\sigma(v)|}{v}\,dv\right)$$

$$+ O\left(\psi(a\sigma)\int_0^a \frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv\right) + O\big(\psi(\sigma)H(\tau_\sigma)\big), \qquad \sigma\to\infty. \tag{38}$$

If

$$\left|\sin\frac{\beta\pi}{2}\right||\tau_\sigma(v)| < |\tau_\sigma(a-v)-\tau_\sigma(a+v)|, \qquad v\in[0,a], \quad a>0. \tag{39}$$

then

$$\mathscr{E}_\sigma\left(\hat{C}^\psi_{\beta,\infty}, U_\sigma(\Lambda)\right)_C = \frac{4}{\pi^2}\psi(a\sigma)\int_0^a \frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv$$

$$+ O\left(\psi(a\sigma)\int_0^{\frac{2}{\sigma\pi}} \frac{j_v|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv\right)$$

$$+ O\left(\psi(\sigma)\left|\sin\frac{\beta\pi}{2}\right|\int_0^\infty \frac{|\tau_\sigma(v)|}{v}\,dv\right) \; O\big(\psi(\sigma)H(\tau_\sigma)\big), \qquad \sigma\to\infty. \tag{40}$$

Proof. Relation (38) follows directly from equality (17) and the definition of the function $\xi(A, B)$. To prove relation (40), note that, by virtue of (39) and (40), the following equalities are true:

$$\int_0^\infty \xi\left(\sin\frac{\beta\pi}{2}\,\tau_\sigma(v),\; j_v[\tau_\sigma(a-v)-\tau_\sigma(a+v)]\right)\frac{dv}{v}$$

$$= \int_0^a\left(\left|\sin\frac{\beta\pi}{2}\right||\tau_\sigma(v)|\arcsin\frac{\left|\sin\frac{\beta\pi}{2}\right||\tau_\sigma(v)|}{|\tau_\sigma(a-v)-\tau_\sigma(a+v)|}\right.$$

$$\left.+\sqrt{(\tau_\sigma(a-v)-\tau_\sigma(a+v))^2-\left(\sin\frac{\beta\pi}{2}\tau_\sigma(v)\right)^2}\right)\frac{dv}{v}$$

$$= \int_0^a\left|\sin\frac{\beta\pi}{2}\right||\tau_\sigma(v)|\left(1-\frac{\left(\sin\frac{\beta\pi}{2}\,\tau_\sigma(v)\right)^2}{(\tau_\sigma(a-v)-\tau_\sigma(a+v))^2}\right)^{\frac{1}{2}}\frac{dv}{v} + O\left(\sin\frac{\beta\pi}{2}\int_0^\infty \frac{|\tau_\sigma(v)|}{v}\,dv\right), \qquad \sigma\to\infty. \tag{41}$$

Since relation (39) is true, we conclude that

$$\frac{\left(\sin\frac{\beta\pi}{2}\tau_\sigma(v)\right)^2}{\left(\tau_\sigma(a-v)-\tau_\sigma(a+v)\right)^2} \in [0,1] \quad \text{if} \quad v\in[0,a].$$

Using relation (41) and the expansion of the function $\sqrt{1-v}$, $v\in[0,1]$, in a power series, we obtain

$$\int_0^\infty \xi\left(\sin\frac{\beta\pi}{2}\tau_\sigma(v),\ j_v[\tau_\sigma(a-v)-\tau_\sigma(a+v)]\right)\frac{dv}{v}$$

$$= \int_0^a \left|\sin\frac{\beta\pi}{2}\right| |\tau_\sigma(v)|\frac{dv}{v} + O\left(\sin\frac{\beta\pi}{2}\int_0^\infty \frac{|\tau_\sigma(v)|}{v}dv\right), \quad \sigma\to\infty. \tag{42}$$

Substituting (42) into (17) and using relation (21), we get (40).

Corollary 1 is proved.

Corollary 2. *Let* $\Lambda=\{\lambda_{n,k}\}$, *where* $n, k = 1, 2, \ldots$ *and* $\lambda_{n,0}=1$ *for all* n, *be a rectangular numerical matrix that associates every function* $f\in\hat{C}^\psi_\beta\mathfrak{N}$ *with series (2). Suppose that the matrix* Λ *is such that series (2) is the Fourier series of a certain continuous function denoted by* $\overline{U}_n(f,x,\Lambda)$. *Also assume that the matrix* Λ *is determined by a sequence of functions* $\lambda_n(u)$, $0\le u<\infty$, *such that* $\lambda_{n,k}=\lambda_n\left(\frac{k}{n}\right)$ *and* $\lambda_{n,0}=1$ *for all* n.

The asymptotic equalities for the quantities

$$\mathcal{E}\left(\hat{C}^\psi_{\beta,\infty},\overline{U}_n(\Lambda)\right)_C = \sup_{f\in\hat{C}^\psi_{\beta,\infty}}\|f(x)-\overline{U}_n(f,x,\Lambda)\|_C$$

can be obtained by setting $\sigma=n$, $n\in N$, *in relations (17), (38), and (40), provided that all conditions of Theorem 1 are satisfied. We get*

$$\mathcal{E}\left(\hat{C}^\psi_{\beta,\infty},\overline{U}_n(\Lambda)\right)_C = \frac{4}{\pi^2}\psi(n)\int_0^\infty\xi\left(\sin\frac{\beta\pi}{2}\tau_n(v),\ j_u[\tau_n(a-v)-\tau_n(a+v)]\right)\frac{dv}{v} + O\big(\psi(\sigma)H(\tau_n)\big), \quad \sigma\to\infty,$$

where the functions $\tau_n(v)$, $n=1,2,\ldots$, *are defined by the equalities*

$$\tau_n(v) = \begin{cases} (1-\lambda_\sigma(v))\dfrac{\psi(1)}{\psi(n)}, & 0\le v\le\dfrac{1}{n},\\[2mm] (1-\lambda_\sigma(v))\dfrac{\psi(nv)}{\psi(n)}, & v\ge\dfrac{1}{n}.\end{cases}$$

If

$$|\tau_n(a-v)-\tau_n(a+v)| \le \left|\sin\frac{\beta\pi}{2}\right| |\tau_n(v)|, \qquad v\in[0,a], \quad a>0,$$

then

$$\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}, \overline{U}_n(\Lambda)\right)_C = \frac{2}{\pi}\psi(n)\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\infty}\frac{|\tau_n(v)|}{v}\,dv + O\left(\psi(n)\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\frac{2}{n\pi}}\frac{|\tau_n(v)|}{v}\,dv\right)$$

$$+ O\left(\psi(an)\int_0^{a}\frac{|\lambda_n(a-v)-\lambda_n(a+v)|}{v}\,dv\right) + O(\psi(n)H(\tau_n)), \qquad \sigma\to\infty.$$

If

$$\left|\sin\frac{\beta\pi}{2}\right| |\tau_n(v)| < |\tau_n(a-v)-\tau_n(a+v)|, \qquad v\in[0,a], \quad a>0,$$

then

$$\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}, \overline{U}_n(\Lambda)\right)_C = \frac{4}{\pi^2}\psi(an)\int_0^{a}\frac{|\lambda_n(a-v)-\lambda_n(a+v)|}{v}\,dv$$

$$+ O\left(\psi(an)\int_0^{\frac{2}{n\pi}}\frac{j_v|\lambda_n(a-v)-\lambda_n(a+v)|}{v}\,dv\right)$$

$$+ O\left(\psi(n)\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\infty}\frac{|\tau_n(v)|}{v}\,dv\right) + O(\psi(n)H(\tau_n)), \qquad n\to\infty.$$

3. Asymptotic Relations for $\mathcal{E}\left(\hat{L}^{\psi}_{\beta,\infty}, U_\sigma(\Lambda)\right)_{\hat{1}}$

In this section, we study the behavior of upper bounds of (6).

First, we give several definitions and auxiliary results.

Assume that $f\in\hat{L}^{\psi}_{\beta}$, $\psi\in F$, and the function $\tau_\sigma(v)$ is defined by (13) and such that its Fourier transform $\hat{\tau}(t)=\hat{\tau}_\sigma(t)$ (14) is summable on R. Then, at almost every point $x\in R$, we have

$$f(x) - U_\sigma(f,x,\Lambda) = \psi(\sigma)\int_{-\infty}^{\infty} f^{\psi}_{\beta}\left(x+\frac{t}{\sigma}\right)\frac{1}{\pi}\int_0^{\infty}\tau_\sigma(v)\cos\left(vt+\frac{\beta\pi}{2}\right)dv\,dt. \tag{43}$$

Note that the function $\tau_\sigma(v)$ can be chosen so that it is continuous for $v\ge 0$ and its Fourier transform $\hat{\tau}_\sigma(t)$ is summable on R. Using relation (43), we establish the following statement:

Lemma 1. *Suppose that* $\psi \in F$, *the function* $\tau_\sigma(\upsilon)$ *is defined by (13) and continuous for all* $\upsilon \geq 0$, *and integral (8) converges. Then the following relation holds as* $\sigma \to \infty$:

$$\mathscr{E}\Big(\hat{L}^{\psi}_{\beta,1}; U_\sigma(\Lambda)\Big)_{\hat{1}} = \sup_{f \in \hat{L}^{\psi}_{\beta,1}} \| f(x) - U_\sigma(f, x, \Lambda) \|_{\hat{1}} = \psi(\sigma) A(\tau_\sigma) + \psi(\sigma)\gamma(\sigma), \tag{44}$$

where $\gamma(\sigma) \leq 0$ *and*

$$|\gamma(\sigma)| = O\left(\int\limits_{|t| \geq \frac{\sigma\pi}{2}} \left| \int\limits_0^\infty \tau_\sigma(\upsilon) \cos\left(\upsilon t + \frac{\beta\pi}{2}\right) d\upsilon \right| dt \right). \tag{45}$$

Proof. Taking into account that

$$\left\| \int\limits_{-\infty}^{\infty} f\left(x + \frac{t}{\sigma}\right) \hat{\tau}_\sigma(t)\, dt \right\|_{\hat{1}} = \sup_{a \in R} \int\limits_{-\pi}^{\pi} \left| \int\limits_{-\infty}^{\infty} f\left(x + a + \frac{t}{\sigma}\right) \hat{\tau}_\sigma(t)\, dt \right| dx \leq \| f \|_{\hat{1}} \int\limits_{-\infty}^{\infty} |\hat{\tau}_\sigma(t)|\, dt$$

and using equalities (6) and (43), we get

$$\mathscr{E}\Big(\hat{L}^{\psi}_{\beta,1}, U_\sigma(\Lambda)\Big)_{\hat{1}} = \sup_{f \in \hat{L}^{\psi}_{\beta,1}} \| f(x) - U_\sigma(f, x, \Lambda) \|_{\hat{1}} \leq \psi(\sigma) \int\limits_{-\infty}^{\infty} |\hat{\tau}_\sigma(t)|\, dt = \psi(\sigma) A(\tau_\sigma). \tag{46}$$

On the other hand, by virtue of Proposition 1.1 in [3, p. 169], we have $\hat{L}^{\psi}_{\beta} L^0_{(0,2\pi)} = L^{\psi}_{\beta}$, where $L^0_{(0,2\pi)}$ is the set of 2π-periodic functions with mean value zero on $(0, 2\pi)$. Therefore, $\hat{L}^{\psi}_{\beta,1} \supset L^{\psi}_{\beta,1}$. Hence,

$$\sup_{f \in \hat{L}^{\psi}_{\beta,1}} \left\| \int\limits_{-\infty}^{\infty} f\left(x + \frac{t}{\sigma}\right) \hat{\tau}_\sigma(t)\, dt \right\|_{\hat{1}} \geq \sup_{f \in L^{\psi}_{\beta,1}} \left\| \int\limits_{-\infty}^{\infty} f\left(x + \frac{t}{\sigma}\right) \hat{\tau}_\sigma(t)\, dt \right\|_1. \tag{47}$$

Moreover, it is shown in [11, p. 41] that

$$\sup_{f \in L^{\psi}_{\beta,1}} \psi(\sigma) \left\| \int\limits_{-\infty}^{\infty} f\left(x + \frac{t}{\sigma}\right) \hat{\tau}_\sigma(t)\, dt \right\|_1 = \psi(\sigma) A(\tau) + \psi(\sigma)\gamma(\sigma), \tag{48}$$

where $\gamma(\sigma) \leq 0$ and relation (45) is true.

Using (46)–(48), we obtain relation (44).

The lemma is proved.

In the periodic case, an analogous lemma was proved in [11] for the classes $L^{\psi}_{\beta,1}$.

Comparing the lemma proved with Lemma 1 in [8], we conclude that the quantities $\mathcal{E}\left(\hat{L}^{\psi}_{\beta,1}, U_{\sigma}(\Lambda)\right)_{\hat{1}}$ and $\mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}, U_{\sigma}(\Lambda)\right)_{C}$ may differ only by a quantity that does not exceed $\gamma(\sigma)$ in order, i.e., the following relation holds as $\sigma \to \infty$:

$$\mathcal{E}\left(\hat{L}^{\psi}_{\beta,1}, U_{\sigma}(\Lambda)\right)_{\hat{1}} = \mathcal{E}\left(\hat{C}^{\psi}_{\beta,\infty}, U_{\sigma}(\Lambda)\right)_{C} + O(\gamma(\sigma)).$$

Using the last equality, we can prove an analog of Theorem 1 for functions of the classes $\hat{L}^{\psi}_{\beta,1}$.

Theorem 2. *Suppose that the following conditions are satisfied:*

(i) $\psi(v) \in F \bigcap \mathfrak{A}_0$;

(ii) $\tau_{\sigma}(v) \in \mathcal{E}_a$;

(iii) $\sin\frac{\beta\pi}{2}\tau_{\sigma}(0) = 0$;

(iv) integrals (15) converge.

Then, for the function $\tau(v) = \tau_{\sigma}(v)$ *defined by (16), the following asymptotic equality is true:*

$$\mathcal{E}\left(\hat{L}^{\psi}_{\beta,1}, U_{\sigma}(\Lambda)\right)_{\hat{1}} = \frac{4}{\pi^2}\psi(\sigma)\int_0^{\infty}\xi\left(\sin\frac{\beta\pi}{2}\tau_{\sigma}(v),\ j_v\left[\tau_{\sigma}(a-v)-\tau_{\sigma}(a+v)\right]\right)\frac{dv}{v}$$

$$+ O\left(\psi(\sigma)\int_0^{\frac{2}{\sigma\pi}}\left(\sin\frac{\beta\pi}{2}\tau_{\sigma}(v) + j_v\left[\tau_{\sigma}(a-v)-\tau_{\sigma}(a+v)\right]\right)\frac{dv}{v}\right)$$

$$+ O\left(\psi(\sigma)H(\tau_{\sigma})\right), \qquad \sigma \to \infty,$$

where $H(\tau_{\sigma})$, $\xi(A, B)$, *and* j_v *are defined by (7), (10), and (11), respectively.*

If, in addition, inequality (37) is true, then

$$\mathcal{E}_{\sigma}\left(\hat{L}^{\psi}_{\beta,1}, U_{\sigma}(\Lambda)\right)_{\hat{1}} = \frac{2}{\pi}\psi(\sigma)\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\infty}\frac{|\tau_{\sigma}(v)|}{v}\,dv + O\left(\psi(\sigma)\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\frac{2}{\sigma\pi}}\frac{|\tau_{\sigma}(v)|}{v}\,dv\right)$$

$$+ O\left(\psi(a\sigma)\int_0^{a}\frac{|\lambda_{\sigma}(a-v)-\lambda_{\sigma}(a+v)|}{v}\,dv\right) + O\left(\psi(\sigma)H(\tau_{\sigma})\right), \qquad \sigma \to \infty.$$

If inequality (39) is true, then

$$\mathscr{E}_\sigma\left(\hat{L}^{\psi}_{\beta,1}, U_\sigma(\Lambda)\right)_{\hat{1}} = \frac{4}{\pi^2}\psi(a\sigma)\int\limits_0^a \frac{|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv$$

$$+ O\left(\psi(a\sigma)\int\limits_0^{\frac{2}{\sigma\pi}} \frac{j_v|\lambda_\sigma(a-v)-\lambda_\sigma(a+v)|}{v}\,dv\right)$$

$$+ O\left(\psi(\sigma)\left|\sin\frac{\beta\pi}{2}\right|\int\limits_0^{\infty}\frac{|\tau_\sigma(v)|}{v}\,dv\right) + O(\psi(\sigma)H(\tau_\sigma)), \qquad \sigma\to\infty.$$

REFERENCES

1. A. I. Stepanets, *Classification and Approximation of Periodic Functions* [in Russian], Naukova Dumka, Kiev (1987).
2. A. I. Stepanets, *Methods of Approximation Theory* [in Russian], Vol. 1, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (2002).
3. A. I. Stepanets, *Methods of Approximation Theory* [in Russian], Vol. 2, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (2002).
4. A. I. Stepanets, "Classes of functions defined on the real axis and their approximation by entire functions. I," *Ukr. Mat. Zh.,* **42**, No. 1, 102–112 (1990).
5. A. I. Stepanets, "Classes of functions defined on the real axis and their approximation by entire functions. II," *Ukr. Mat. Zh.,* **42**, No. 2, 210–222 (1990).
6. M. G. Dzimistarishvili, *Approximation of Classes of Continuous Functions by Zygmund Operators* [in Russian], Preprint No. 89.25, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1989), pp. 3–42.
7. M. G. Dzimistarishvili, *On the Behavior of Upper Bounds of Deviations of Steklov Operators* [in Russian], Preprint No. 90.25, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1990), pp. 3–29.
8. V. I. Rukasov, "Approximation of functions defined on the real axis by de la Vallée-Poussin operators," *Ukr. Mat. Zh.,* **44**, No. 5, 682–691 (1992).
9. L. A. Repeta, "Approximation of functions of the classes $\hat{C}^{\psi}_{\beta,\infty}$ by operators of the form $U^{\varphi,F}_{\sigma}$," in: *Fourier Series: Theory and Applications* [in Russian], Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1992), pp. 147–154.
10. L. I. Bausov, "Linear methods of summation of Fourier series with given rectangular matrices. I," *Izv. Vyssh. Uchebn. Zaved.,* **46**, No. 3, 15–31 (1965).
11. A. K. Novikova, "On approximation of functions in the spaces C and L," in: *Problems of Summation of Fourier Series* [in Russian], Preprint No. 85.61, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1985), pp. 14–51.
12. S. A. Telyakovskii, "On the norms of trigonometric polynomials and approximation of differentiable functions by linear means of their Fourier series. II," *Izv. Akad. Nauk SSSR, Ser. Mat.,* **27**, No. 2, 253–272 (1963).
13. A. I. Stepanets, "Approximation of functions with slowly varying Fourier coefficients by Fourier sums," in: *Approximation of Periodic Functions by Fourier Sums* [in Russian], Preprint No. 84.43, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1984), pp. 3–25.