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APPROXIMATION OF DIFFERENTIABLE PERIODIC FUNCTIONS BY THEIR BIHARMONIC POISSON INTEGRALS

K. M. Zhyhallo and Yu. I. Kharkevych

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We determine the exact values and asymptotic decompositions of upper bounds of approximations by biharmonic Poisson integrals on classes of periodic differentiable functions.

Let f be a 2π-periodic function summable on $[-\pi, \pi]$. The function

$$P_\rho(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left\{\frac{1}{2} + \sum_{k=1}^{\infty}\left[1 + \frac{k}{2}(1-\rho^2)\right]\rho^k \cos kt\right\}dt, \qquad 0 \le \rho < 1, \quad -\pi \le x < \pi,$$

is called the biharmonic Poisson integral of the function f.

Let W^r, $r \in N$, be the set of 2π-periodic functions having absolutely continuous derivatives up to the $(r-1)$th order inclusive and such that $\operatorname*{ess\,sup}_{x \in R} |f^r(x)| \le 1$, and let $\overline{W}^r$ be the class of functions conjugate to functions from the class W^r, i.e.,

$$\overline{W}^r = \left\{\bar{f} : \bar{f}(x) := -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\operatorname{ctg}\frac{t}{2}\,dt, \ f \in W^r\right\}. \tag{1}$$

Denote

$$\mathcal{E}(\mathfrak{N}, P_\rho)_C = \sup_{f \in \mathfrak{N}} \|f(x) - P_\rho(f, x)\|_C, \qquad 0 \le \rho < 1, \tag{2}$$

where $\|f\|_C = \max_{t \in R} |f(t)|$.

If a function $g(\rho) = g(\mathfrak{N}; \rho)$ such that

$$\mathcal{E}(\mathfrak{N}, P_\rho)_C = g(\rho) + o(g(\rho)) \qquad \text{as} \quad \rho \to 1-$$

has been found in explicit form, then, following Stepanets [1, pp. 67–68], we say that the Kolmogorov–Nikol'skii problem is solved for the given class $\mathfrak{N}$ and approximating aggregate P_ρ.

A formal series $\sum_{n=0}^{\infty} g_n(\rho)$ is called (see, e.g., [2, p. 21]) an asymptotic decomposition of a function $f(\rho)$ as $\rho \to 1-$ if, for all n, we have

$$|g_{n+1}(\rho)| = o(|g_n(\rho)|)$$

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and, for any natural N,

$$f(\rho) = \sum_{n=0}^{N} g_n(\rho) + o(g_N(\rho)), \qquad \rho \to 1-.$$

We briefly write this fact as follows:

$$f(\rho) \cong \sum_{n=0}^{\infty} g_n(\rho).$$

Quantities (2) and their analogs with the harmonic Poisson integral

$$A_\rho(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left\{\frac{1}{2} + \sum\nolimits_{k=1}^{\infty}\rho^k \cos kt\right\}dt$$

instead of $P_\rho(f, x)$ were studied by Timan [3], Sz.-Nagy [4], Kaniev [5], Shtark [6], Zhyhallo and Kharkevych [7–9], Falaleev [10], etc.

The aim of the present work is to determine the exact value of quantity (2) for $\mathfrak{N} = \overline{W}^r$, $r \in N \setminus \{1\}$, and find its asymptotic decomposition for $\mathfrak{N} = W^r$ and $\mathfrak{N} = \overline{W}^r$, $r \in N \setminus \{1\}$.

Theorem 1. *For any* $l \in N$ *and* $0 < \rho < 1$, *the following equalities are true:*

$$\mathcal{E}\left(\overline{W}^{2l}, P_\rho\right)_C = \sum_{i=1}^{l} \frac{1}{(2i-1)!} K_{2(l-i)+1} \ln^{2i-1}\frac{1}{\rho} - \sum_{i=1}^{l-1} \frac{1}{(2i)!} \tilde{K}_{2(l-i)} \ln^{2i}\frac{1}{\rho}$$
$$+ \frac{1-\rho^2}{2}\left[\sum_{i=1}^{l-1} \frac{1}{(2i-1)!} \tilde{K}_{2(l-i)} \ln^{2i-1}\frac{1}{\rho} - \sum_{i=0}^{l-1} \frac{1}{(2i)!} K_{2(l-i)-1} \ln^{2i}\frac{1}{\rho}\right] - \varepsilon_\rho^{2l} + \frac{1-\rho^2}{2}\varepsilon_\rho^{2l-1}, \quad (3)$$

$$\varepsilon_\rho^r = \frac{2}{\pi}\int_\rho^1\int_{t_r}^1 \ldots \int_{t_2}^1 \frac{1}{t_1 \ldots t_r} \ln\frac{1+t_1}{1-t_1}\, dt_1 \ldots dt_r;$$

$$\mathcal{E}\left(\overline{W}^{2l+1}, P_\rho\right)_C = \sum_{i=1}^{l} \frac{1}{(2i-1)!} K_{2(l-i)+2} \ln^{2i-1}\frac{1}{\rho} - \sum_{i=1}^{l} \frac{1}{(2i)!} \tilde{K}_{2(l-i)+1} \ln^{2i}\frac{1}{\rho}$$
$$+ \frac{1-\rho^2}{2}\left[\sum_{i=1}^{l} \frac{1}{(2i-1)!} \tilde{K}_{2(l-i)+1} \ln^{2i-1}\frac{1}{\rho} - \sum_{i=0}^{l-1} \frac{1}{(2i)!} K_{2(l-i)} \ln^{2i}\frac{1}{\rho}\right] + \delta_\rho^{2l+1} - \frac{1-\rho^2}{2}\delta_\rho^{2l}, \quad (4)$$

$$\delta_\rho^r = \frac{4}{\pi}\int_\rho^1\int_{t_r}^1 \ldots \int_{t_2}^1 \frac{1}{t_1 \ldots t_r} \arctan t_1 dt_1 \ldots dt_r,$$

where K_n *and* $\tilde{K}_n$ *are the Akhiezer–Krein–Favard constants:*

$$K_n = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^{m(n+1)}}{(2m+1)^{n+1}}, \quad n = 0, 1, 2, \ldots, \qquad \tilde{K}_n = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^{mn}}{(2m+1)^{n+1}}, \quad n \in N.$$

In the proof of this theorem, we follow the scheme of the proof of Theorems 1 and 2 in [9].

Taking into account the integral representation (1) and the fact that

$$P_\rho(\bar{f},\varphi) = \bar{P}_\rho(f,\varphi) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\sum_{k=1}^{\infty}\left[1+\frac{k}{2}(1-\rho^2)\right]\rho^k \sin kt\,dt$$

for $f \in W^r$, $r \in N\backslash\{1\}$, and integrating r times by parts, we get

$$\mathcal{E}\left(\overline{W}^r, P_\rho\right)_C = \frac{1}{\pi}\sup_{f\in W^r}\left|\int_{-\pi}^{\pi} f^{(r)}(t)\overline{V}_r(\rho,t)\,dt\right|, \tag{5}$$

where

$$\overline{V}_r(\rho,t) = \sum_{k=1}^{\infty}\frac{1-\left[1+\frac{k}{2}(1-\rho^2)\right]\rho^k}{k^r}\cos\left(kt+\frac{(r+1)\pi}{2}\right), \quad 0 \le \rho < 1.$$

If we prove that, for $l \in N$, the equalities

$$\operatorname{sign}\overline{V}_{2l}(\rho,t) = \pm\operatorname{sign}\sin t \tag{6}$$

and

$$\operatorname{sign}\left(\overline{V}_{2l+1}(\rho,t)-\overline{V}_{2l+1}\left(\rho,\frac{\pi}{2}\right)\right) = \pm\operatorname{sign}\cos t \tag{7}$$

are true, then relation (5) will yield

$$\mathcal{E}\left(\overline{W}^r, P_\rho\right)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^{kr}\frac{1-\left[1+\frac{2k+1}{2}(1-\rho^2)\right]\rho^{2k+1}}{(2k+1)^{r+1}}. \tag{8}$$

To prove (6) and (7), we first show that

$$\overline{V}_2(\rho,t) = \sum_{k=1}^{\infty} c_k \sin kt > 0, \qquad c_k := \frac{1-\left[1+\frac{k}{2}(1-\rho^2)\right]\rho^k}{k^2}$$

for $t \in (0,\pi)$ and $0 \le \rho < 1$. For this purpose, we analyze the sequence of the coefficients c_k of the decomposition of this kernel. Since

$$\Delta c_k = c_k - c_{k+1} = \frac{1}{k^2} - \frac{1}{(k+1)^2} - \frac{\rho^k}{k^2} + \frac{\rho^{k+1}}{(k+1)^2} - \frac{1-\rho^2}{2}\left(\frac{\rho^k}{k} - \frac{\rho^{k+1}}{k+1}\right) =: \xi_k(\rho),$$

$\xi_k(0) > 0$, $\xi_k(1) = 0$, and

$$\xi'(\rho) = (1-\rho^2)\rho^{k-1}\left(\frac{\rho}{k+1} - \frac{1}{k} + \frac{\rho-1}{2}\right) < 0, \quad 0 \le \rho < 1, \quad k = 1, 2, \ldots,$$

we get $\Delta c_k > 0$ for any $k \in N$ and $0 \le \rho < 1$. Taking into account that

$$\Delta^2 c_k = c_k - 2c_{k+1} + c_{k+2}$$

$$= \frac{1}{k^2} + \frac{1}{(k+2)^2} - \frac{2}{(k+1)^2} - \frac{\rho^k}{k^2} - \frac{\rho^{k+2}}{(k+2)^2} + \frac{2\rho^{k+1}}{(k+1)^2} - \frac{1-\rho^2}{2}\left(\frac{\rho^k}{k} + \frac{\rho^{k+2}}{k+2} - \frac{2\rho^{k+1}}{k+1}\right) =: \eta_k(\rho)$$

and, for any $k \in N$,

$$\eta_k(0) > 0, \qquad \eta_k(1) = 0,$$

$$\eta_k'(\rho) = (1-\rho^2)\rho^{k-1}\left(-\frac{\rho^2}{k+2} - \frac{1}{k} + \frac{2\rho}{k+1} - \frac{(1-\rho)^2}{2}\right) < 0, \quad 0 \le \rho < 1,$$

we conclude that $\Delta^2 c_k > 0$ for all $k \in N$ and $0 \le \rho < 1$. Note that the coefficients c_k of the kernel $\overline{V}_2(t) = \overline{V}_2(\rho, t)$ are positive and tend to zero twice monotonically $(\Delta c_k > 0,\ \Delta^2 c_k > 0)$. Furthermore, it is easy to see that they satisfy the condition $\sum_{k=1}^{\infty} \frac{c_k}{k} < \infty$. Therefore, according to [11, pp. 297–298], we get $\overline{V}_2(\rho, t) > 0$ for $t \in (0, \pi)$ and $0 \le \rho < 1$.

It is obvious that $\overline{V}_{2l}(\rho, 0) = \overline{V}_{2l}(\rho, \pi) = 0$ for $l \in N$. Hence, assuming that $\overline{V}_{2l}(\rho, t) = 0$ for one more point $t_0 \in (0, \pi)$ and using the Rolle theorem $2l - 2$ times, we establish that, for the function $\overline{V}_2(\rho, t)$, there exists $t_{2l-2} \in (0, \pi)$ such that $\overline{V}_2(\rho, t_{2l-2}) = 0$. However, as shown above, this is impossible. Thus, equality (6) is true. Further, if we assume that $\overline{V}_{2l+1}(\rho, t_0) - \overline{V}_{2l+1}\left(\rho, \frac{\pi}{2}\right) = 0$, $t_0 \in (0, \pi)$, $t_0 \ne \frac{\pi}{2}$, then, according to the Rolle theorem, there exists $t_1 \in (0, \pi)$ such that $\overline{V}_{2l+1}'(\rho, t_1) = 0$, whence $\overline{V}_{2l}'(\rho, t_1) = 0$. But this is impossible by virtue of (6). Equality (7) is proved.

Finally, using relation (8), we get

$$\mathscr{E}\left(\overline{W}^r, P_\rho\right)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^{kr}\frac{1-\rho^{2k+1}}{(2k+1)^{r+1}} - \frac{2(1-\rho^2)}{\pi}\sum_{k=0}^{\infty}(-1)^{kr}\frac{1}{(2k+1)^r} + \frac{2(1-\rho^2)}{\pi}\sum_{k=0}^{\infty}(-1)^{kr}\frac{1-\rho^{2k+1}}{(2k+1)^r}. \quad (9)$$

Using relation (9) and taking into account that the functions

$$\varphi_n(\rho) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-\rho^{2k+1}}{(2k+1)^{n+1}},$$

$$\psi_n(\rho) = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1-\rho^{2k+1}}{(2k+1)^{n+1}}$$

satisfy the equalities (see [3, p. 20])

$$\varphi_n(\rho) = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} \varphi_{n-k}(0) \ln^k \frac{1}{\rho} + (-1)^{n-1} \varepsilon_\rho^n,$$

$$\psi_n(\rho) = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!} \psi_{n-k}(0) \ln^k \frac{1}{\rho} + (-1)^{n-1} \delta_\rho^n,$$

we obtain the statement of Theorem 1.

Remark 1. Note that, by virtue of (8), we have

$$\mathscr{E}\left(\overline{W}^r, P_0\right)_C = \tilde{K}_r, \quad r = 2, 3, \ldots .$$

Remark 2. Taking into account that

$$\delta_\rho^r = O((1-\rho)^r), \qquad \varepsilon_\rho^r = O\left((1-\rho)^r \ln \frac{1}{1-\rho}\right) \tag{10}$$

as $\rho \to 1-$ (see [3, p. 18]), by virtue of Theorem 1 we get

$$\mathscr{E}\left(\overline{W}^2, P_\rho\right)_C = O\left((1-\rho)^2 \ln \frac{1}{1-\rho}\right) \tag{11}$$

and

$$\mathscr{E}\left(\overline{W}^r, P_\rho\right)_C = \left(K_{r-1} + \frac{\tilde{K}_{r-2}}{2}\right)(1-\rho)^2 + O\left((1-\rho)^3 \ln \frac{1}{1-\rho}\right), \qquad r = 3, 4, \ldots, \tag{12}$$

as $\rho \to 1-$. Comparing (11) and (12) with the estimates

$$\mathscr{E}\left(\overline{W}^r, A_\rho\right)_C = K_{r-1}(1-\rho) + O\left((1-\rho)^2 \ln \frac{1}{1-\rho}\right) \tag{13}$$

obtained in [9] (here, A_ρ is the harmonic Poisson integral), we conclude that, in the case where $r \in N \setminus \{1\}$ and $\rho \to 1-$, the right-hand sides of (11) and (12) are smaller by an order of magnitude than the right-hand side of (13).

In view of relation (10) and the estimate $\ln \frac{1}{\rho} \sim (1-\rho)$ as $\rho \to 1-$, equalities (3) for $r = 2l$ and (4) for $r = 2l+1$ enable one to specify only the first $r-1$ terms of asymptotics with the corresponding constants (Kolmogorov–Nikol'skii constants). The theorems presented below give asymptotic decompositions of quantities (2) for $\mathfrak{N} = W^r$ and $\mathfrak{N} = \overline{W}^r$ that enable one to calculate the Kolmogorov–Nikol'skii constants corresponding to asymptotic terms of arbitrarily high order of smallness. In the proof of these theorems, we use the following lemmas from [8] (for technical reasons independent of the authors, several misprints were made in their formulation in [8]):

Lemma 1. *For the functions*

$$\varphi_n(\rho) = \int_\rho^1 \int_0^{t_n} \ldots \int_0^{t_2} \frac{1}{t_1 \ldots t_n} \ln \frac{1+t_1}{1-t_1} dt_1 \ldots dt_n, \qquad n \in N,$$

the following asymptotic decomposition is true:

$$\varphi_n(\rho) \cong \sum_{k=1}^{\infty} \left\{ \alpha_k^n (1-\rho)^k \ln \frac{1}{1-\rho} + \beta_k^n (1-\rho)^k \right\},$$

where

$$\alpha_k^n = \frac{(-1)^k}{k!} a_n^k, \tag{14}$$

$$\beta_k^n = \frac{(-1)^k}{k!} \left\{ \sum_{i=1}^{n-1} \varphi_{n-i}(0) a_i^k + a_n^k \left(\ln 2 + \sum_{i=1}^{k} \frac{1}{i} \right) + S_k^n \right\}, \tag{15}$$

$$S_k^n = \begin{cases} 0, & k \le n; \\ \sum\limits_{i=n+1}^{k} \dfrac{a_i^k}{2^{i-n}} + \sum\limits_{i=1}^{k-n} A_i^{k-1} a_n^{k-i}, & k > n, \end{cases}$$

$$a_i^j = \begin{cases} 0, & i > j; \\ (-1)^j (j-1)!, & i = 1; \\ a_{i-1}^{j-1} - a_i^{j-1}(j-1), & i \le j \le n; \\ a_{i-1}^{j-1} - a_i^{j-1}(j-2), & n+1 = i \le j; \\ -(i-n-1) a_{i-1}^{j-1} - a_i^{j-1}(j-i+n-1), & n+1 < i \le j, \end{cases}$$

$$A_k^n = \frac{n(n-1)\ldots(n-k+1)}{k}, \qquad \varphi_n(0) = \begin{cases} \dfrac{\pi}{2} K_n, & n \text{ is odd,} \\ \dfrac{\pi}{2} \tilde{K}_n, & n \text{ is even.} \end{cases}$$

Lemma 2. *For the functions*

$$\psi_n(\rho) = \int_\rho^1 \int_0^{t_n} \ldots \int_0^{t_2} \frac{\arctan t_1}{t_1 \ldots t_n} dt_1 \ldots dt_n, \qquad n \in N,$$

the following asymptotic decomposition is true:

$$\psi_n(\rho) = \sum_{k=1}^{\infty} \gamma_k^n (1-\rho)^k,$$

where, for $k \in N$,

$$\gamma_k^n = \frac{(-1)^k}{k!}\left\{\sum_{i=1}^{n}\psi_{n-i}(0)b_i^k + \sigma_k^n\right\}, \tag{16}$$

$$\psi_n(0) = \begin{cases} \dfrac{\pi}{4}K_n, & n \text{ is odd}, \\ \dfrac{\pi}{4}\tilde{K}_n, & n \text{ is even}, \end{cases} \qquad \sigma_k^n = \begin{cases} 0, & k \le n, \\ \displaystyle\sum_{i=n+1}^{k}\frac{b_i^k}{2^{i-n}}, & k > n, \end{cases}$$

$$b_i^j = \begin{cases} 0, & i > j, \\ (-1)^j(j-1)!, & i = 1, \\ b_{i-1}^{j-1} - b_i^{j-1}(j-1), & i \le j \le n, \\ b_{i-1}^{j-1} - b_i^{j-1}(j-2), & n+1 = i \le j, \\ -2(i-n-1)b_{i-1}^{j-1} - b_i^{j-1}(j-2i+2n), & n+1 < i \le j. \end{cases}$$

Theorem 2. *The following asymptotic decompositions are true:*

$$\mathcal{E}\left(\overline{W}^2, P_\rho\right)_C \cong \frac{1}{\pi}(1-\rho)^2\ln\frac{1}{1-\rho} + \left(K_1 + \frac{\ln 2}{\pi} - \frac{1}{2\pi}\right)(1-\rho)^2$$
$$+ \frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\left[\alpha_k^2 + \alpha_{k-1}^1 - \frac{1}{2}\alpha_{k-2}^1\right](1-\rho)^k\ln\frac{1}{1-\rho} + \left[\beta_k^2 + \beta_{k-1}^1 - \frac{1}{2}\beta_{k-2}^1\right](1-\rho)^k\right\},$$

$$\mathcal{E}\left(\overline{W}^r, P_\rho\right)_C \cong \left(K_{r-1} + \frac{1}{2}\tilde{K}_{r-2}\right)(1-\rho)^2 + \frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\left[\alpha_k^r + \alpha_{k-1}^{r-1} - \frac{1}{2}\alpha_{k-2}^{r-1}\right](1-\rho)^k\ln\frac{1}{1-\rho}\right.$$
$$\left. + \left[\beta_k^r + \beta_{k-1}^{r-1} - \frac{1}{2}\beta_{k-2}^{r-1}\right](1-\rho)^k\right\}, \quad r = 2l+2, \quad l \in N,$$

$$\mathcal{E}\left(\overline{W}^r, P_\rho\right)_C \cong \left(K_{r-1} + \frac{1}{2}\tilde{K}_{r-2}\right)(1-\rho)^2 + \frac{4}{\pi}\sum_{k=3}^{\infty}\left[\gamma_k^r + \gamma_{k-1}^{r-1} - \frac{1}{2}\gamma_{k-2}^{r-1}\right](1-\rho)^k, \quad r = 2l+1, \quad l \in N,$$

where the coefficients α_k^r, β_k^r, *and* γ_k^r *are calculated according to formulas (14), (15), and (16), respectively.*

The validity of Theorem 2 follows from equality (9), Lemmas 1 and 2, and the relations

$$\int_{\rho}^{1}\int_{0}^{t_n}\ldots\int_{0}^{t_2}\frac{1}{t_1\ldots t_n}\ln\frac{1+t_1}{1-t_1}dt_1\ldots dt_n = 2\sum_{k=0}^{\infty}\frac{1-\rho^{2k+1}}{(2k+1)^{n+1}} \tag{17}$$

and

$$\int_{\rho}^{1}\int_{0}^{t_n}\ldots\int_{0}^{t_2}\frac{\arctan t_1}{t_1\ldots t_n}dt_1\ldots dt_n = \sum_{k=0}^{\infty}(-1)^k\frac{1-\rho^{2k+1}}{(2k+1)^{n+1}}. \tag{18}$$

Theorem 3. *The following asymptotic decompositions are true:*

$$\mathcal{E}(W^r, P_\rho)_C \cong \left(\tilde{K}_{r-1}+\frac{1}{2}K_{r-2}\right)(1-\rho)^2+\frac{2}{\pi}\sum_{k=3}^{\infty}\left\{\left[\alpha_k^r+\alpha_{k-1}^{r-1}-\frac{1}{2}\alpha_{k-2}^{r-1}\right](1-\rho)^k\ln\frac{1}{1-\rho}\right.$$

$$\left.+\left[\beta_k^r+\beta_{k-1}^{r-1}-\frac{1}{2}\beta_{k-2}^{r-1}\right](1-\rho)^k\right\}, \quad r=2l+1, \quad l\in N, \tag{19}$$

$$\mathcal{E}(W^r, P_\rho)_C \cong \left(\tilde{K}_{r-1}+\frac{1}{2}K_{r-2}\right)(1-\rho)^2+\frac{4}{\pi}\sum_{k=3}^{\infty}\left[\gamma_k^r+\gamma_{k-1}^{r-1}-\frac{1}{2}\gamma_{k-2}^{r-1}\right](1-\rho)^k, \quad r=2l, \quad l\in N, \tag{20}$$

where the coefficients α_k^r, β_k^r, *and* γ_k^r *are calculated according to formulas (14), (15), and (16), respectively.*

Proof. As in the proof of Theorem 1, it is easy to show that

$$\mathcal{E}(W^r, P_\rho)_C = \frac{1}{\pi}\sup_{f\in W^r}\left|\int_{-\pi}^{\pi}f^{(r)}(t)V_r(\rho,t)\,dt\right|,$$

where

$$V_r(\rho,t) = \sum_{k=1}^{\infty}\frac{1-\left[1+\frac{k}{2}(1-\rho^2)\right]\rho^k}{k^r}\cos\left(kt+\frac{r\pi}{2}\right),$$

and, furthermore,

$$\operatorname{sign}V_r(\rho,t) = \pm\operatorname{sign}\sin t \quad \text{for} \quad r=2l+1,$$

$$\operatorname{sign}\left(V_r(\rho,t)-V_r\left(\rho,\frac{\pi}{2}\right)\right) = \pm\operatorname{sign}\cos t \quad \text{for} \quad r=2l.$$

Therefore, for $r\ge 2$, we get

$$\mathcal{E}(W^r, P_\rho)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^{k(r+1)}\frac{1-\left[1+\frac{2k+1}{2}(1-\rho^2)\right]\rho^{2k+1}}{(2k+1)^{r+1}}$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^{k(r+1)}\frac{1-\rho^{2k+1}}{(2k+1)^{r+1}}-\frac{2(1-\rho^2)}{\pi}\sum_{k=0}^{\infty}(-1)^{k(r+1)}\frac{1}{(2k+1)^r}$$

$$+\frac{2(1-\rho^2)}{\pi}\sum_{k=0}^{\infty}(-1)^{k(r+1)}\frac{1-\rho^{2k+1}}{(2k+1)^r}. \tag{21}$$

Taking into account relations (17) and (18) and Lemmas 1 and 2, we obtain the asymptotic decompositions (19) and (20) from (21). Theorem 3 is proved.

Note that asymptotic decompositions of the upper bounds of approximations on the class of differentiable functions W^1 by their harmonic Poisson integrals were obtained in [6], and on the classes W^r, $r \in N \backslash \{1\}$, and $\overline{W}^r$, $r \in N$, in [8]. An asymptotic decomposition of $\mathscr{E}(W^1, P_\rho)_C$ was obtained in [7].

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