Ukrainian Mathematical Journal, Vol. 54, No. 1, 2002

COMPLETE ASYMPTOTICS OF THE DEVIATION OF A CLASS OF DIFFERENTIABLE FUNCTIONS FROM THE SET OF THEIR HARMONIC POISSON INTEGRALS

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UDC 517.5

On a class of differentiable functions W^r and the class $\overline{W}^r$ of functions conjugate to them, we obtain a complete asymptotic expansion of the upper bounds $\mathcal{E}(\mathfrak{N}, A_\rho)_C$ of deviations of the harmonic Poisson integrals of the functions considered.

1. Statement of the Problem and Auxiliary Statements

Let W^r, $r \in N$, be the set of 2π-periodic functions that have absolutely continuous derivatives up to the $(r-1)$th order inclusive and are such that $\operatorname*{ess\,sup}_{x \in R} |f^r(x)| \le 1$. Let $\overline{W}^r$ denote the class of functions conjugate to functions from the class W^r, i.e.,

$$\overline{W}^r = \left\{ \bar{f} : \bar{f}(x) := -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \cot\frac{t}{2} dt, \ f \in W^r \right\}.$$

For a 2π-periodic function f summable over the period, let $A_\rho(f, x)$ and $\overline{A}_\rho(f, x)$ denote the harmonic Poisson integral and the conjugate harmonic Poisson integral, respectively, i.e.,

$$A_\rho(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) P(\rho, t)\, dt, \qquad 0 \le \rho < 1,$$

where

$$P(\rho, t) = \frac{1}{2} \frac{1-\rho^2}{1 - 2\rho\cos t + \rho^2} = \frac{1}{2} + \sum_{k=1}^{\infty} \rho^k \cos kt$$

is the Poisson kernel, and

$$\overline{A}_\rho(f, x) = A_\rho(\bar{f}, x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) Q(\rho, t)\, dt, \tag{1}$$

where

Volyn University, Luts'k. Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 54, No. 1, pp. 43–52, January, 2002. Original article submitted May 25, 2001.

$$Q(\rho, t) = \frac{\rho \sin t}{1 - 2\rho \cos t + \rho^2} = \sum_{k=1}^{\infty} \rho^k \sin kt \tag{2}$$

is the conjugate Poisson kernel.

It is known (see, e.g., [1, Chap. 1]) that if f is continuous on R, then, for every $x \in R$, we have $\lim\limits_{\rho \to 1-} \overline{A}_\rho(f; x) = \bar{f}(x)$.

In the present paper, we study the behavior of the quantity

$$\mathcal{E}(\mathfrak{N}, A_\rho)_C = \sup_{f \in \mathfrak{N}} \| f(x) - A_\rho(f, x) \|_C$$

as $\rho \to 1-$ in the cases $\mathfrak{N} \equiv W^r$ and $\mathfrak{N} \equiv \overline{W}^r$ (here, $\| f \|_C = \max\limits_{x \in R} |f(x)|$).

If there exists an explicit function $g(\rho) = g(\mathfrak{N}; \rho)$ such that

$$\mathcal{E}(\mathfrak{N}, A_\rho)_C = g(\rho) + o(g(\rho)) \tag{3}$$

as $\rho \to 1-$, then it is said [2] that the Kolmogorov – Nikol'skii problem is solved for a given class $\mathfrak{N}$ and an approximating aggregate A_ρ.

A formal series $\sum\limits_{n=0}^{\infty} g_n(\rho)$ is called a *complete asymptotic expansion,* or *complete asymptotics* [3], of a function $f(\rho)$ as $\rho \to 1-$ if, for all n, we have

$$|g_{n+1}(\rho)| = o(|g_n(\rho)|) \tag{4}$$

and, for any natural N,

$$f(\rho) = \sum_{n=0}^{N} g_n(\rho) + o(g_N(\rho)), \quad \rho \to 1-. \tag{5}$$

We denote this as follows:

$$f(\rho) \cong \sum_{n=0}^{\infty} g_n(\rho).$$

Natanson [4] obtained the first result of the form (3) for $\mathfrak{N} = W^1$:

$$\mathcal{E}(W^1, A_\rho)_C = \frac{2}{\pi}(1-\rho)|\ln(1-\rho)| + O(1-\rho). \tag{6}$$

Timan [5] improved this result (he obtained the second asymptotic term) and generalized it to the classes W^r, $r \in N$.

For the classes $\mathfrak{N} = \overline{W}^1$, Sz.-Nagy [6] established the asymptotic equality

$$\mathscr{E}(\overline{W}^1, A_\rho)_C = (1-\rho) + O((1-\rho)^2), \quad \rho \to 1-. \tag{7}$$

In equalities (6) and (7), only the first terms of the asymptotics are given with the corresponding constants.

In the present paper, we obtain complete asymptotic expansions (in the above sense) for the quantities $\mathscr{E}(W^r, A_\rho)_C$, and $\mathscr{E}(\overline{W}^r, A_\rho)_C$, $r \in N$.

Let K_n and $\tilde{K}_n$ denote the well-known Akhiezer–Krein–Favard constants, i.e.,

$$K_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m(n+1)}}{(2m+1)^{n+1}}, \quad n \ge 0, \qquad \tilde{K}_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{mn}}{(2m+1)^{n+1}}, \quad n \ge 1.$$

First, we establish facts concerning the complete asymptotic expansion of the special functions

$$\varphi_n(\rho) = \int_\rho^1\int_0^{t_n}\cdots\int_0^{t_2}\frac{1}{t_1\ldots t_n}\ln\frac{1+t_1}{1-t_1}\,dt_1\ldots dt_n, \tag{8}$$

$$\psi_n(\rho) = \int_\rho^1\int_0^{t_n}\cdots\int_0^{t_2}\frac{\arctan t_1}{t_1\ldots t_n}\,dt_1\ldots dt_n, \quad n \in N, \tag{9}$$

in terms of which the estimates of the quantities $\mathscr{E}(W^r, A_\rho)_C$ and $\mathscr{E}(\overline{W}^r, A_\rho)_C$ are expressed.

Lemma 1. *For the functions* $\varphi_n(\rho)$, $n \in N$, *the following complete asymptotic expansion is true:*

$$\varphi_n(\rho) \cong \sum_{k=1}^{\infty}\left\{\alpha_k^n(1-\rho)^k\ln\frac{1}{1-\rho} + \beta_k^n(1-\rho)^k\right\},$$

where, for $k \in N$,

$$\alpha_k^n = \frac{(-1)^k}{k!}a_n^k, \tag{10}$$

$$\beta_k^n = \frac{(-1)^k}{k!}\left\{\sum_{i=1}^{n-1}\varphi_{n-i}(0)a_i^k + a_n^k\left(\ln 2 + \sum_{i=1}^{k}\frac{1}{i}\right) + S_k^n\right\}, \tag{11}$$

$$S_k^n = \begin{cases} 0, & k \le n, \\ \displaystyle\sum_{i=n+1}^{k}\frac{a_i^k}{2^{i-n}} + \sum_{i=1}^{k-n}A_i^{k-1}a_n^{k-i}, & k > n, \end{cases}$$

$$a_i^j \;=\; \begin{cases} 0, & i > j, \\ (-1)^j (j-1)!, & i = 1, \\ a_{i-1}^{j-1} - a_i^{j-1}(j-1), & i \le j \le n, \\ a_{i-1}^{j-1} - a_i^{j-1}(j-2), & n+1 = i \le j, \\ -(i-n-1)\,a_{i-1}^{j-1} - a_i^{j-1}(j-i+n-1), & n+1 < i \le j, \end{cases} \tag{12}$$

$$A_k^n \;=\; \frac{n(n-1)\ldots(n-k+1)}{k}, \qquad \varphi_n(0) \;=\; \begin{cases} \dfrac{\pi}{2K_n}, & n \ \textit{is odd}, \\ \dfrac{\pi}{2\tilde{K}_n}, & n \ \textit{is even}. \end{cases}$$

Proof. First, note that, in a complete asymptotic expansion of the form

$$\varphi_n(\rho) \;\cong\; \sum_{k=1}^{\infty}\left\{\alpha_k^n(1-\rho)^k \ln\frac{1}{1-\rho} + \beta_k^n(1-\rho)^k\right\}, \tag{13}$$

the coefficients α_k^n and β_k^n must satisfy the following relations:

$$\alpha_k^n \;=:\; \lim_{\rho\to 1-}\frac{-1}{(1-\rho)^k\ln(1-\rho)}\left\{\varphi_n(\rho) + \sum_{j=1}^{k-1}\left[\alpha_j^n(1-\rho)^j\ln(1-\rho) - \beta_j^n(1-\rho)^j\right]\right\}, \tag{14}$$

$$\beta_k^n \;=:\; \lim_{\rho\to 1-}\frac{1}{(1-\rho)^k}\left\{\varphi_n(\rho) + \alpha_k^n(1-\rho)^k\ln(1-\rho) \;+\; \sum_{j=1}^{k-1}\left[\alpha_j^n(1-\rho)^j\ln(1-\rho) - \beta_j^n(1-\rho)^j\right]\right\} \tag{15}$$

[to verify conditions (4) and (5), one must set $g_{2k-1} = \alpha_k^n(1-\rho)^k\ln\dfrac{1}{1-\rho}$ and $g_{2k} = \beta_k^n(1-\rho)^k$].

Hence, to prove Lemma 1, it suffices to show that the coefficients α_k^n and β_k^n determined from (14) and (15) have the forms (10) and (11), respectively.

Applying the l'Hospital rule k times to indeterminacies of the type $0/0$, for $k = 1$ and $n > 1$ we get

$$\alpha_1^n \;=\; \lim_{\rho\to 1-}\frac{-\varphi_n(\rho)}{(1-\rho)\ln(1-\rho)} \;=\; \lim_{\rho\to 1-}\frac{-1}{(1+\ln(1-\rho))\rho}\int_0^{\rho}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\frac{1}{t_1\ldots t_{n-1}}\ln\frac{1+t_1}{1-t_1}\,dt_1\ldots dt_{n-1} \;=\; 0,$$

$$\beta_1^n \;=\; \lim_{\rho\to 1-}\frac{\varphi_n(\rho)}{1-\rho} \;=\; \lim_{\rho\to 1-0}\frac{1}{\rho}\int_0^{r}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\frac{1}{t_1\ldots t_{n-1}}\ln\frac{1+t_1}{1-t_1}\,dt_1\ldots dt_{n-1} \;=\; \varphi_{n-1}(0).$$

For $k \le n-1$, taking into account that

$$\frac{d^k\varphi_n(\rho)}{d\rho^k} = \frac{a_1^k}{\rho^k}\int_0^{\rho}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\frac{1}{t_1\ldots t_{n-1}}\ln\frac{1+t_1}{1-t_1}dt_1\ldots dt_{n-1}$$

$$+ \frac{a_2^k}{\rho^k}\int_0^{\rho}\int_0^{t_{n-2}}\cdots\int_0^{t_2}\frac{1}{t_1\ldots t_{n-2}}\ln\frac{1+t_1}{1-t_1}dt_1\ldots dt_{n-2} + \ldots + \frac{a_{n-1}^k}{\rho^k}\int_0^{\rho}\frac{1}{t_1}\ln\frac{1+t_1}{1-t_1}dt_1$$

$$+ \frac{a_n^k}{\rho^k}\ln\frac{1+\rho}{1-\rho} + \frac{a_{n+1}^k}{\rho^{k-1}}\left(\frac{1}{1+\rho}+\frac{1}{1-\rho}\right) + \frac{a_{n+2}^k}{\rho^{k-2}}\left(\frac{1}{(1+\rho)^2}-\frac{1}{(1-\rho)^2}\right)$$

$$+ \ldots + \frac{a_k^k}{\rho^n}\left(\frac{1}{(1+\rho)^{k-n}}+\frac{(-1)^{k-n+1}}{(1-\rho)^{k-n}}\right), \tag{16}$$

where the coefficients a_i^k, $i = \overline{1, k}$, are successively determined from the recurrence relations (12), and

$$\frac{d^k}{d\rho^k}\left((1-\rho)^k\ln(1-\rho)\right) = (-1)^k k!\left(\ln(1-\rho)+\sum_{i=1}^{k}\frac{1}{i}\right), \tag{17}$$

we obtain

$$\alpha_k^n = 0,$$

$$\beta_k^n = \frac{1}{(-1)^k k!}\sum_{i=1}^{k}\varphi_{n-i}(0)a_i^k.$$

In the case $k = n$, by using relation (17), we obtain

$$\alpha_n^n = \lim_{\rho\to 1-}\frac{-d^n\varphi_n(\rho)/d\rho^n}{(-1)^n n!\left(\ln(1-\rho)+\sum_{i=1}^{n}\frac{1}{i}\right)},$$

$$\beta_n^n = \frac{1}{(-1)^n n!}\lim_{\rho\to 1-}\left(\frac{d^n\varphi_n(\rho)}{d\rho^n}+\alpha_n^n(-1)^n n!\left(\ln(1-\rho)+\sum_{i=1}^{n}\frac{1}{i}\right)\right).$$

Hence, according to (16), we have

$$\alpha_n^n = \frac{a_n^n}{(-1)^n n!},$$

$$\beta_n^n = \frac{1}{(-1)^n n!}\left(\sum_{i=1}^{n-1}\varphi_{n-i}(0)a_i^n + a_n^n\left(\ln 2+\sum_{i=1}^{n}\frac{1}{i}\right)\right).$$

Now consider the case $k > n$. For this purpose, we use the relations

$$\frac{d^k}{d\rho^k}\{(1-\rho)^{\mu}\ln(1-\rho)\} = \frac{(-1)^{\mu+1}\mu!\,(k-\mu-1)!}{(1-\rho)^{k-\mu}}, \qquad k > \mu \ge 0, \tag{18}$$

which follows from (17), and

$$\lim_{\rho\to 1-}\sum_{i=1}^{k-n}\left(\frac{a_{i+n}^k(-1)^{i+1}}{\rho^{k-i}} - a_n^{k-i}(i-1)!\right)\frac{1}{(1-\rho)^i} = \sum_{i=1}^{k-n} A_i^{k-1}a_n^{k-i}, \tag{19}$$

where a_i^j are satisfy relations (12). To establish (19), it suffices to pass to the limit as $\rho \to 1-$ in the relation

$$\sum_{i=1}^{k-n}\left(\frac{a_{i+n}^k(-1)^{i+1}}{\rho^{k-i}} - a_n^{k-i}(i-1)!\right)\frac{1}{(1-\rho)^i}$$

$$= \sum_{i_{k-n}=1}^{k-1}\left(a_n^{k-1}\,0! - \sum_{i_{k-n-1}=1}^{i_{k-n}-1}\left(a_n^{k-2}\,1! - \ldots - \sum_{i_1=1}^{i_2-1}a_n^n(k-n-1)!\right)\right)\ldots\Bigg)\rho^{i_{k-n}-1}$$

and take into account that

$$\sum_{i_1=1}^{k-1}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_m=1}^{i_{m-1}-1}1 = \frac{(k-1)(k-2)\ldots(k-m)}{m!}, \qquad m \le k-1.$$

By using relations (14), (15), (18), and (19), we obtain the following formulas for the calculation of α_k^n and β_k^n for $k > n$:

$$\alpha_k^n = \frac{a_n^k}{(-1)^k\,k!}$$

and

$$\beta_k^n = \lim_{\rho\to 1-}\frac{1}{(-1)^k\,k!}\left\{\frac{a_1^k}{\rho^k}\int_0^{\rho}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\frac{1}{t_1\ldots t_{n-1}}\ln\frac{1+t_1}{1-t_1}\,dt_1\ldots dt_{n-1}\right.$$

$$+ \frac{a_2^k}{\rho^k}\int_0^{\rho}\int_0^{t_{n-2}}\cdots\int_0^{t_2}\frac{1}{t_1\ldots t_{n-2}}\ln\frac{1+t_1}{1-t_1}\,dt_1\ldots dt_{n-2} + \ldots + \frac{a_{n-1}^k}{\rho^k}\int_0^{\rho}\frac{1}{t_1}\ln\frac{1+t_1}{1-t_1}\,dt_1$$

$$+ \frac{a_n^k}{\rho^k}\ln\frac{1+\rho}{1-\rho} + \frac{a_{n+1}^k}{\rho^{k-1}}\left(\frac{1}{1+\rho} + \frac{1}{1-\rho}\right) + \frac{a_{n+2}^k}{\rho^{k-2}}\left(\frac{1}{(1+\rho)^2} - \frac{1}{(1-\rho)^2}\right)$$

$$\left. + \ldots + \frac{a_k^k}{\rho^n}\left(\frac{1}{(1+\rho)^{k-n}} + \frac{(-1)^{k-n+1}}{(1-\rho)^{k-n}}\right) - \sum_{j=n}^{k-1}a_n^j\frac{(k-j-1)!}{(1-\rho)^{k-j}} + a_n^k\left(\ln(1-\rho) + \sum_{i=1}^{k}\frac{1}{i}\right)\right\}$$

$$= \frac{1}{(-1)^k k!}\left\{\sum_{i=1}^{n-1}\varphi_{n-i}(0)a_i^k + a_n^k\left(\ln 2 + \sum_{i=1}^{k}\frac{1}{i}\right) + \sum_{i=n+1}^{k}\frac{a_i^k}{2^{i-n}} + \sum_{i=1}^{k-n}A_i^{k-1}a_n^{k-i}\right\}.$$

Lemma 1 is proved.

Lemma 2. *For the functions* $\psi_n(\rho)$, $n \in N$, *the following complete asymptotic expansion is true:*

$$\psi_n(\rho) = \frac{4}{\pi}\sum_{k=1}^{\infty}\gamma_k^n(1-\rho)^k,$$

where, for $k \in N$,

$$\gamma_k^n = \frac{(-1)^k}{k!}\left\{\sum_{i=1}^{n}\psi_{n-i}(0)b_i^k + \sigma_k^n\right\}, \tag{20}$$

$$\psi_n(0) = \begin{cases} \frac{\pi}{4}K_n, & n \text{ is even}, \\ \frac{\pi}{4}\tilde{K}_n, & n \text{ is odd}, \end{cases} \qquad \sigma_k^n = \begin{cases} 0, & k \le n, \\ \sum_{i=n+1}^{k}\frac{b_i^k}{2^{i-n}}, & k > n, \end{cases}$$

$$b_i^j = \begin{cases} 0, & i > j, \\ (-1)^j(j-1)!, & i = 1, \\ b_{i-1}^{j-1} - b_i^{j-1}(j-1), & i \le j \le n, \\ b_{i-1}^{j-1} - b_i^{j-1}(j-2), & n+1 = i \le j, \\ -2(i-n-1)b_{i-1}^{j-1} - b_i^{j-1}(j-2i+2n), & n+1 < i \le j. \end{cases} \tag{21}$$

Proof. In a complete asymptotic expansion of the form

$$\psi_n(\rho) \cong \sum_{k=1}^{\infty}\gamma_k^n(1-\rho)^k,$$

the coefficients γ_k^n must satisfy the following relations:

$$\gamma_k^n =: \lim_{r\to 1-}\frac{1}{(1-\rho)^k}\left\{\psi_n(\rho) - \sum_{j=1}^{k-1}\gamma_j^n(1-\rho)^j\right\}. \tag{22}$$

Hence, to prove Lemma 2, it suffices to show that the coefficients γ_k^n determined from (22) have the form (20).

Applying the l'Hospital rule k times to indeterminacies of the form $0/0$ and using the fact that

$$\frac{d^k\psi_n(\rho)}{d\rho^k} = \frac{b_1^k}{\rho^k}\int_0^{\rho}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\frac{\arctan t_1}{t_1\ldots t_{n-1}}dt_1\ldots dt_{n-1}$$

$$+ \frac{b_2^k}{\rho^k}\int_0^{\rho}\int_0^{t_{n-2}}\cdots\int_0^{t_2}\frac{\arctan t_1}{t_1\ldots t_{n-2}}dt_1\ldots dt_{n-2} + \ldots + \frac{b_{n-1}^k}{\rho^k}\int_0^{\rho}\frac{\arctan t_1}{t_1}dt_1$$

$$+ \frac{b_n^k}{\rho^k}\arctan\rho + \frac{b_{n+1}^k}{\rho^{k-1}}\frac{1}{1+\rho^2} + \frac{b_{n+2}^k}{\rho^{k-2}}\frac{1}{(1+\rho^2)^2} + \ldots + \frac{b_k^k}{\rho^{k-2(k-n)+1}}\frac{1}{(1+\rho^2)^{k-n}},$$

where the coefficients b_i^k, $i=\overline{1,k}$, are successively determined from the recurrence relations (21), we obtain

$$\gamma_k^n = \lim_{\rho\to 1-}\frac{1}{(-1)^k(k)!}\left\{\frac{b_1^k}{\rho^k}\int_0^{\rho}\int_0^{t_{n-1}}\cdots\int_0^{t_2}\frac{\arctan t_1}{t_1\ldots t_{n-1}}dt_1\ldots dt_{n-1}\right.$$

$$+ \frac{b_2^k}{\rho^k}\int_0^{\rho}\int_0^{t_{n-2}}\cdots\int_0^{t_2}\frac{\arctan t_1}{t_1\ldots t_{n-2}}dt_1\ldots dt_{n-2} + \ldots + \frac{b_{n-1}^k}{\rho^k}\int_0^{\rho}\frac{\arctan t_1}{t_1}dt_1$$

$$\left. + \frac{b_n^k}{\rho^k}\arctan r + \frac{b_{n+1}^k}{\rho^{k-1}}\frac{1}{1+\rho^2} + \frac{b_{n+2}^k}{\rho^{k-2}}\frac{1}{(1+\rho^2)^2} + \frac{b_k^k}{\rho^{k-2(k-n)+1}}\frac{1}{(1+\rho^2)^{k-n}}\right\}.$$

Lemma 2 is proved.

2. Main Results for Classes $\overline{W}^r$

Theorem 1. *If* $r=2l$, $l\in N$, *then the following complete asymptotic expansion is true:*

$$\mathcal{E}(\overline{W}^r, A_\rho)_C \cong \frac{2}{\pi}\sum_{k=1}^{\infty}\left\{\alpha_k^r(1-\rho)^k\ln\frac{1}{1-r} + \beta_k^r(1-\rho)^k\right\}, \tag{23}$$

where the coefficients α_k^r *and* β_k^r *are determined from relations (10)–(12).*

Proof. Taking into account relations (1) and (2), we obtain

$$\bar{f}(x) - A_\rho(\bar{f}, x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\left\{\frac{1}{2}\cot\frac{t}{2} - \sum_{k=1}^{\infty}\rho^k\sin kt\right\}dt.$$

Hence, integrating r times by parts, we get

$$\bar{f}(x) - A_\rho(\bar{f}, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t+x)\sum_{k=1}^{\infty}\frac{1-\rho^k}{k^r}\cos\left[kt + \frac{(r+1)\pi}{2}\right]dt.$$

Therefore,

$$\mathscr{E}(\overline{W}^r, A_\rho)_C = \frac{1}{\pi}\sup_{f\in W^r}\left|\int_{-\pi}^{\pi} f^{(r)}(t)\,\overline{F}_{r,\rho}(t)\,dt\right|,$$

where

$$\overline{F}_{r,\rho}(t) = \sum_{k=1}^{\infty}\frac{1-\rho^k}{k^r}\cos\left(kt + \frac{(r+1)\pi}{2}\right).$$

Since $f\in W^r$ and $F_{r,\rho}(t)$ is odd for $r = 2l$, $l\in N$, we have

$$\mathscr{E}(\overline{W}^r, A_\rho)_C \le \frac{2}{\pi}\int_0^{\pi}\left|\overline{F}_{r,\rho}(t)\right|dt.$$

On the other hand, if $\operatorname{sign}\overline{F}_{r,\rho}(t) = \pm\operatorname{sign}\sin t$, then a function f such that

$$f^{(r)}(t) = \operatorname{sign}(\overline{F}_{r,\rho}(t)), \qquad t\in[-\pi,\pi],$$

is continuously and periodically extendable to R and belongs to the class W^r [7, pp. 104–106]. Therefore, for $r = 2l$, $l\in N$, we have

$$\mathscr{E}(\overline{W}^r, A_\rho)_C \ge \frac{2}{\pi}\int_0^{\pi}\left|\overline{F}_{r,\rho}(t)\right|dt$$

and, hence,

$$\mathscr{E}(\overline{W}^r, A_\rho)_C = \frac{2}{\pi}\int_0^{\pi}\left|\overline{F}_{r,\rho}(t)\right|dt = \frac{2}{\pi}\left|\int_0^{\pi}\overline{F}_{r,\rho}(t)\,dt\right|. \tag{24}$$

The fact that, for $r = 2l$, $l\in N$, the function $\overline{F}_{r,\rho}(t)$ changes its sign on $(0,\pi)$ [i.e., $\operatorname{sign}\overline{F}_{r,\rho}(t) = \pm\operatorname{sign}\sin t$] is established by the following reasoning:

It is clear that, for $r = 2l$, $l\in N$, we have $\overline{F}_{r,\rho}(0) = \overline{F}_{r,\rho}(\pi) = 0$. Therefore, under the assumption that $\overline{F}_{r,\rho}(t) = 0$ for some $t_0\in(0,\pi)$, by virtue of the Rolle theorem there exist $t_0^{(1)}\in(0,t_0)$ and $t_0^{(2)}\in(t_0,\pi)$ such that $\overline{F}'_{r,\rho}(t_0^{(1)}) = \overline{F}'_{r,\rho}(t_0^{(2)}) = 0$. Hence,

$$\overline{F}_{r-1,\rho}(t_0^{(1)}) = \overline{F}_{r-1,\rho}(t_0^{(2)}) = 0$$

and, consequently, there exists $t_1 \in (t_0^1, t_0^2)$ such that $\overline{F}'_{r-1,\rho}(t_1) = 0$, i.e., $\overline{F}_{r-2,\rho}(t_1) = 0$. Therefore, by analogy with the arguments presented above, we conclude that there exist $t_1^{(1)} \in (0, t_1)$ and $t_1^{(2)} \in (t_1, \pi)$ such that

$$\overline{F}_{r-3,\rho}(t_1^{(1)}) = \overline{F}_{r-3,\rho}(t_1^{(2)}) = 0,$$

and so on. Repeating this procedure as many times as necessary, we establish that, under the original assumption concerning the function

$$\overline{F}_{1,\rho}(t) = -\sum_{k=1}^{\infty} \frac{1-\rho^k}{k} \cos kt,$$

there exist $t_{l-1}^{(1)}, t_{l-1}^{(2)} \in (0, \pi)$, $t_{l-1}^{(1)} \neq t_{l-1}^{(2)}$, such that

$$\overline{F}_{1,\rho}(t_{l-1}^{(1)}) = \overline{F}_{1,\rho}(t_{l-1}^{(2)}) = 0.$$

However, this contradicts the fact that, according to relations (1.441.2) and (1.448.2) in [8], the function $\overline{F}_{1,\rho}(t)$ can be represented in the form

$$\overline{F}_{1,\rho}(t) = \frac{1}{2} \ln \frac{2(1-\cos t)}{1 - 2\rho\cos t + \rho^2}, \qquad t \in (0, \pi),$$

and, as can easily be verified, the equation $\overline{F}_{1,\rho}(t) = 0$ has only one root on the interval $(0, \pi)$.

Thus, by using relation (24), for $r = 2l$, $l \in N$, we obtain

$$\mathcal{E}(\overline{W}^r, A_\rho)_C = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1-\rho^{2k+1}}{(2k+1)^{r+1}}.$$

Hence, taking into account that [5]

$$2 \sum_{k=0}^{\infty} \frac{1-\rho^{2k+1}}{(2k+1)^{n+1}} = \varphi_n(\rho),$$

where $\varphi_n(\rho)$ is the function defined by (8), and using Lemma 1, we obtain the statement of Theorem 1.

Remark 1. For $r = 1$, expansion (23) is a refined version of the asymptotic equality (7).

Theorem 2. *If* $r = 2l - 1$, $l \in N$, *then the following complete asymptotic expansion is true:*

$$\mathcal{E}(\overline{W}^r, A_\rho)_C \cong \frac{4}{\pi} \sum_{k=1}^{\infty} \gamma_k^r (1-\rho)^k,$$

where the coefficients γ_k^r *are determined from relations (20) and (21).*

Proof. According to [6, p. 187], we have

$$\mathscr{E}(\overline{W}^1, A_\rho)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{1-\rho^{2k+1}}{(2k+1)^2}.$$

Let $r = 2l + 1,\ l \in N$. Then, by analogy with the proof of Theorem 1, one can show that

$$\mathscr{E}(\overline{W}^r, A_\rho)_C = \frac{1}{\pi}\sup_{f\in W^r}\left|\int_{-\pi}^{\pi} f^{(r)}(t)\overline{F}_{r,\rho}(t)dt\right| = \frac{1}{\pi}\sup_{f\in W^r}\left|\int_{-\pi}^{\pi} f^{(r)}(t)\left(\overline{F}_{r,\rho}(t)-\overline{F}_{r,\rho}\left(\frac{\pi}{2}\right)\right)dt\right|.$$

Since $f \in W^r$ and $\overline{F}_{r,\rho}(t)$ is even for $r = 2l + 1,\ l \in N$, we have

$$\mathscr{E}(\overline{W}^r, A_\rho)_C \le \frac{2}{\pi}\int_0^{\pi}\left|\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}\left(\frac{\pi}{2}\right)\right|dt.$$

On the other hand, if $\operatorname{sign}(\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}(\pi/2)) = \pm\operatorname{sign}\cos t$, then a function f such that $f^{(r)}(t) = \operatorname{sign}(\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}(\pi/2)),\ t \in [-\pi, \pi]$, is continuously and periodically extendable to R and belongs to the class W^r [7, pp. 187–188]. Therefore, for $r = 2l + 1,\ l \in N$, we have

$$\mathscr{E}(\overline{W}^r, A_\rho)_C \ge \frac{2}{\pi}\int_0^{\pi}\left|\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}\left(\frac{\pi}{2}\right)\right|dt$$

and, hence,

$$\begin{aligned}\mathscr{E}(\overline{W}^r, A_\rho)_C &= \frac{2}{\pi}\int_0^{\pi}\left|\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}\left(\frac{\pi}{2}\right)\right|dt \\ &= \frac{2}{\pi}\left|\int_0^{\pi/2}\left(\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}\left(\frac{\pi}{2}\right)\right)dt - \int_0^{\pi/2}\left(\overline{F}_{r,\rho}(\pi - t) - \overline{F}_{r,\rho}\left(\frac{\pi}{2}\right)\right)dt\right| \\ &= \frac{2}{\pi}\left|\int_0^{\pi/2}(\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}(\pi - t))\,dt\right|. \qquad (25)\end{aligned}$$

The equality $\operatorname{sign}(\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}(\pi/2)) = \pm\operatorname{sign}\cos t$ is established by the following reasoning:

Under the assumption that $\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}(\pi/2) = 0,\ r = 2l + 1,\ l \in N$, for some $t_0 \in (0, \pi),\ t_0 \ne \pi/2$, by virtue of the Rolle theorem there exists $t_0^{(1)} \in (0, \pi)$ such that $\overline{F}'_{r,\rho}(t_0^{(1)}) = 0$, whence

$$\overline{F}_{r-1,\rho}(t_0^{(1)}) = 0.$$

However, this contradicts the fact that $\operatorname{sign}\overline{F}_{r-1,\rho}(t) = \pm\operatorname{sign}\sin t$ for $r = 2l+1$, $l \in N$. Consequently, $t = \pi/2$ is a unique solution of the equation $\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}(\pi/2) = 0$ on the segment $[0, \pi]$. Furthermore, since $\operatorname{sign}\overline{F}'_{r,\rho}(t) = \pm\operatorname{sign}\sin t$ for $r = 2l+1$, $l \in N$, the function $\overline{F}_{r,\rho}(t) - \overline{F}_{r,\rho}(\pi/2)$ is monotone on $(0, \pi)$.

Hence, by using relation (25), for $r = 2l+1$, $l \in N$, we get

$$\mathcal{E}(\overline{W}^r, A_\rho)_C = \left| \int_0^{\pi/2} \sum_{k=0}^{\infty} \frac{1-\rho^{2k+1}}{(2k+1)^r} \cos(2k+1)t\,dt \right|.$$

Thus, for $r = 2l-1$, $l \in N$, we have

$$\mathcal{E}(\overline{W}^r, A_\rho)_C = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1-\rho^{2k+1}}{(2k+1)^{r+1}}.$$

Hence, taking into account that [5]

$$\sum_{k=0}^{\infty} (-1)^k \frac{1-\rho^{2k+1}}{(2k+1)^{n+1}} = \psi_n(\rho),$$

where $\psi_n(\rho)$ is the function defined by (9), and using Lemma 2, we obtain the statement of Theorem 2.

3. Statement for Classes W^r

Theorem 3. *The following complete asymptotic expansions are true:*

$$\mathcal{E}(W^r, A_\rho)_C \cong \begin{cases} \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \left\{ \alpha_k^r (1-\rho)^k \ln \frac{1}{1-r} + \beta_k^r (1-\rho)^k \right\}, & r = 2l-1,\ l \in N, \\ \dfrac{4}{\pi} \displaystyle\sum_{k=1}^{\infty} \gamma_k^r (1-\rho)^k, & r = 2l,\ l \in N, \end{cases}$$

where the coefficients α_k^r *and* β_k^r *are determined from relations (10)–(12), and the coefficients* γ_k^r *are determined from relations (20) and (21).*

Proof. According to relation (7) in [5], we have

$$\mathcal{E}(W^r, A_\rho)_C = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^{k(r+1)} \frac{1-\rho^{2k+1}}{(2k+1)^{r+1}}. \tag{26}$$

For $r = 2l$, $l \in N$, the right-hand side of equality (26) identically coincides with the function $\psi_r(\rho)$, $0 \le \rho < 1$ [see (8)]; for $r = 2l-1$, $l \in N$, it identically coincides with the function $\varphi_r(\rho)$, $0 \le \rho < 1$ [see (9)].

The complete asymptotic expansions of these functions are given in Lemmas 1 and 2, respectively. Theorem 3 is proved.

Remark 2. In the case $r = 1$, the statement of Theorem 3 was established in [9].

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