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ON THE APPROXIMATION OF FUNCTIONS OF THE HÖLDER CLASS BY TRIHARMONIC POISSON INTEGRALS

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We determine the exact value of the upper bound for the deviation of the triharmonic Poisson integral from functions of the Hölder class.

Let $L^1_{2\pi}$ and $C_{2\pi} = L^\infty_{2\pi}$ denote the classes of 2π-periodic integrable and continuous functions, respectively, with the norms

$$\|f\|_1 = \int_{-\pi}^{\pi} |f(x)|\,dx, \qquad \|f\|_\infty = \max_x |f(x)|.$$

We denote by

$$A_2(r,\theta) = \int_{-\pi}^{\pi} f(t+\theta) P_2(r,t)\,dt,$$

where $f(\cdot) \in L_{2\pi}$ and

$$P_2(r,t) = \frac{(1-r^2)^2(1-r\cos t)}{2\pi(1-2r\cos t + r^2)^2}, \qquad 0 \le r < 1,$$

the biharmonic function that is called the biharmonic Poisson integral [1, 2].

Correspondingly, by

$$A_3(r,\theta) = \int_{-\pi}^{\pi} f(t+\theta) P_3(r,t)\,dt,$$

where $f(\cdot) \in L_{2\pi}$ and

$$P_3(r,t) = \frac{(1-r^2)^3(4 - 9r\cos t + 6r^2\cos^2 t - r^3\cos t)}{8\pi(1-2r\cos t + r^2)^3}, \qquad 0 \le r < 1,$$

we denote, according to [3–5], the triharmonic function that is called the triharmonic Poisson integral.

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The set of functions $f \in L^p_{2\pi}$, $p = 1, \infty$, satisfying the inequality

$$\| f(x+h) - f(x) \|_p \le |h|, \tag{1}$$

is denoted by H^1_p and is called the Hölder class.

The set of functions $f \in L^p_{2\pi}$, $p = 1, \infty$, satisfying the inequality

$$\| f(t+h) - 2f(t) + f(t-h) \|_p \le 2|h|$$

is denoted by H^2_p and is called the class of quasismooth functions [6].

Let

$$\mathcal{E}\left(H^\nu_p, A_n(r,\theta)\right)_p = \sup_{f \in H^\nu_p} \| A_n(r,\theta) - f(\theta) \|_p, \tag{2}$$

where $p = 1, \infty$, $\nu = 1, 2$, and $n = 2, 3$, denote the upper bound of the deviation of the functions of the class H^ν_p from their biharmonic and triharmonic Poisson integrals.

If a function

$$\varphi(1-r) = \varphi\left(H^\nu_p; A_n(r,\theta); (1-r)\right)$$

such that, as $r \to 1-0$,

$$\mathcal{E}\left(H^\nu_p, A_n(r,\theta)\right)_p = \varphi(1-r) + o(\varphi(1-r))$$

is determined in explicit form, then one says that the Kolmogorov–Nikol'skii problem is solved [7] for a given operator $A_n(r,\theta)$ and a given class H^ν_p, $p = 1, \infty$, $\nu = 1, 2$.

A formal series $\sum_{n=0}^\infty \varphi_n(r)$ is called an asymptotic expansion of the function f as $r \to r_0$ if, for all n, we have $|\varphi_{n+1}(r)| = o(|\varphi_n(r)|)$ as $r \to r_0$ and, furthermore, for any $N < \infty$,

$$f(r) - \sum_{n=0}^{N} \varphi_n(r) = O\left(\varphi_{N+1}(r)\right), \qquad r \to r_0.$$

In 1963, Kaniev [8] proved the following asymptotic equality for the quantity $\mathcal{E}\left(H^2_\infty, A_2(r,\theta)\right)_\infty$ as $r \to 1-0$:

$$\mathcal{E}\left(H^2_\infty, A_2(r,\theta)\right)_\infty = \frac{2}{\pi}(1-r) + \frac{\varepsilon_r}{\pi}, \qquad \text{where} \quad \varepsilon_r = o(1-r). \tag{3}$$

In 1968, Pych [9] established the asymptotic equality

$$\mathcal{E}\Big(H^2_\infty, A_2(r,\theta)\Big)_\infty = \frac{2}{\pi}(1-r) + O\left((1-r)^2 \ln\frac{1}{1-r}\right), \qquad r \to 1-0. \tag{4}$$

Estimates (3) and (4) enable one to determine the first asymptotic constant (the Kolmogorov–Nikol'skii constant) (see [10]) in the approximation of functions of the class H^2_∞ by their biharmonic Poisson integrals. It should be noted that $\mathcal{E}\Big(H^1_p, A_2(r,\theta)\Big)_p = \mathcal{E}\Big(H^2_p, A_2(r,\theta)\Big)_p$, $p = 1, \infty$ (see, e.g., [11]).

In [12], an asymptotic expansion was obtained for $\mathcal{E}\Big(H^1_p, A_2(r,\theta)\Big)_p$, $p = 1, \infty$, which enables one to successively determine the Kolmogorov–Nikol'skii constants of arbitrary order of smallness.

In the present paper, we establish the exact value of the upper bound of the deviation of the triharmonic Poisson integral from functions of the Hölder class H^1_p, $p = 1, \infty$.

The following statement is true:

Theorem 1. *The quantity* $\mathcal{E}\Big(H^1_p, A_3(r,\theta)\Big)_p$ *defined by equality (2) has the following asymptotic expansion as* $r \to 1-0$:

$$\mathcal{E}\Big(H^1_p, A_3(r,\theta)\Big)_p = \frac{1}{\pi}(1-r) + \frac{5}{2\pi}(1-r)^2 + \frac{8}{3\pi}(1-r)^3 \ln\frac{1}{1-r}$$

$$+ \frac{1}{\pi}\left(\frac{1}{18} + \frac{8}{3}\ln 2\right)(1-r)^3 - \frac{1}{\pi}(1-r)^4 + \frac{1}{\pi}\sum_{k=5}^{\infty}\left\{\frac{2}{k}(1-r)^k \ln\frac{1}{1-r} + \gamma_k (1-r)^k\right\}, \tag{5}$$

where $p = 1, \infty$ *and*

$$\gamma_k = \frac{2}{k}\left(\ln 2 + \frac{1}{k} - \sum_{j=1}^{k-1}\frac{2^{-j}}{j}\right) + \frac{1}{2^{k-2}}\left\{\frac{1}{k-1} + \frac{1}{k-2} - \frac{4}{k-3} + \frac{2}{k-4}\right\}.$$

Proof. First, we prove equality (5) for $p = \infty$. For this purpose, we set

$$P(r,t) = \frac{1-r^2}{1-2r\cos t + r^2}.$$

Since

$$P_3(r,t) = \frac{3(1-r^2)^2}{16\pi}P(r,t) + \frac{(3-5r^2)(1-r^2)}{16\pi}P^2(r,t) + \frac{(1-r^2)^2}{8\pi}P^3(r,t)$$

$$=: P_3'(r,t) + P_3''(r,t) + P_3'''(r,t), \tag{6}$$

one can easily verify that

$$\int_{-\pi}^{\pi} P_3(r,t)\,dt = 1.$$

Then, obviously,

$$A_3(r,\theta) - f(\theta) = \int_{-\pi}^{\pi} \{ f(t+\theta) - f(\theta) \} P_3(r,t)\,dt.$$

Since $f \in H^1$, by virtue of relation (1) we obtain

$$|A_3(r,\theta) - f(\theta)| \le \int_{-\pi}^{\pi} |t| P_3(r,t)\,dt.$$

It follows from the fact that the function $f_0(x) = |x|$, $x \in [-\pi,\pi]$, belongs to H^1 and turns the above inequality into the equality, we obtain

$$\mathcal{E}\left(H^1_\infty, A_3(r,\theta)\right)_\infty = \sup_{f \in H^1} \max_{|\theta| \le \pi} |A_3(r,\theta) - f(\theta)| = 2\int_0^{\pi} t\,P_3(r,t)\,dt. \tag{7}$$

Taking into account relation (6) and using (7), we get

$$\mathcal{E}\left(H^1_\infty, A_3(r,\theta)\right)_\infty = 2\int_0^{\pi} tP_3'(r,t)\,dt + 2\int_0^{\pi} tP_3''(r,t)\,dt + 2\int_0^{\pi} tP_3'''(r,t)\,dt =: I_1(r) + I_2(r) + I_3(r).$$

To calculate the integral $I_2(r)$, we perform integration by parts. For this purpose, we set $u = t$ and $dv = dt \,/\, (1 - 2r\cos t + r^2)^2$. Using formulas 2.554 (3) and 2.553 (3) from [13], we obtain

$$v = \frac{1}{(1-r^2)^2}\frac{2r\sin t}{1-2r\cos t + r^2} + \frac{2(1+r^2)}{(1-r^2)^3}\arctan\left(\frac{1+r}{1-r}\tan\frac{t}{2}\right).$$

Then

$$I_2(r) = \frac{(1-r^2)^3(3-5r^2)}{8\pi}\left(\pi\frac{1+r^2}{(1-r^2)^3}\pi - \frac{2r}{(1-r^2)^2}\int_0^{\pi}\frac{\sin t\,dt}{1-2r\cos t + r^2}\right.$$

$$\left. - \frac{2(1+r^2)}{(1-r^2)^3}\int_0^{\pi}\arctan\left(\frac{1+r}{1-r}\,tan\,\frac{t}{2}\right)dt\right).$$

Hence,

$$I_2(r) = \frac{(1+r^2)(3-5r^2)}{8\pi}\int_0^{\pi} t\frac{1-r^2}{1-2r\cos t+r^2}dt - \frac{(1-r^2)(3-5r^2)}{4\pi}\ln\frac{1+r}{1-r}.$$

Similarly, by using formulas 2.554 (3) from [13], we obtain

$$I_3(r) = -\frac{(1-r^2)r}{2\pi} - \frac{3(1+r^2)(1-r^2)}{4\pi}\ln\frac{1+r}{1-r} + \frac{(1+r^2)^2+2r^2}{4\pi}\int_0^{\pi} t\frac{1-r^2}{1-2r\cos t+r^2}dt.$$

Taking into account the equalities obtained for $I_2(r)$ and $I_3(r)$, we get

$$\mathcal{E}\left(H_\infty^1, A_3(r,\theta)\right)_\infty = I_1(r) + I_2(r) + I_3(r) =: \frac{1}{\pi}(l_1(r) + l_2(r) + l_3(r)), \tag{8}$$

where

$$l_1(r) = -\frac{r(1-r^2)}{2}, \qquad l_2(r) = -\frac{(1-r^2)(3-r^2)}{2}\ln\frac{1+r}{1-r},$$

$$l_3(r) = \int_0^{\pi} t\frac{1-r^2}{1-2r\cos t+r^2}dt.$$

We expand the function $l_1(r)$ in a Taylor series in powers of $r-1$. As a result, we obtain

$$l_1(r) = -(1-r) + \frac{3}{2}(1-r)^2 - \frac{1}{2}(1-r)^3. \tag{9}$$

Since

$$\ln(1+r) = \ln 2 - \sum_{k=1}^{\infty}\frac{(1-r)^k}{k2^k},$$

we have

$$l_2(r) = -2\ln 2(1-r) + (1-\ln 2)(1-r)^2 + \left(2\ln 2 + \frac{3}{4}\right)(1-r)^3$$
$$+ \left(-\frac{\ln 2}{2} - \frac{19}{24}\right)(1-r)^4 - 2(1-r)\ln\frac{1}{1-r} - (1-r)^2\ln\frac{1}{1-r} + 2(1-r)^3\ln\frac{1}{1-r}$$
$$- \frac{1}{2}(1-r)^4\ln\frac{1}{1-r} + \sum_{k=5}^{\infty}\alpha_k(1-r)^k, \tag{10}$$

where

$$\alpha_k = \frac{1}{2^{k-2}}\left(\frac{1}{k-1} + \frac{1}{k-2} - \frac{4}{k-3} + \frac{2}{k-4}\right).$$

Taking into account the asymptotic expansion as $r \to 1-0$ obtained by Shtark in [14, p. 23], we get

$$l_3(r) = 2\sum_{k=1}^{\infty}\left\{\frac{1}{k}(1-r)^k \ln\frac{1}{1-r} + \beta_k(1-r)^k\right\}, \tag{11}$$

where

$$\beta_k = \frac{1}{k}\left\{\ln 2 + \frac{1}{k} - \sum_{j=1}^{k-1}\frac{2^{-j}}{j}\right\}.$$

Substituting (9)–(11) in (8) and performing identity transformations, we establish equality (5) in the case $p = \infty$.

In the case $p = 1$, relation (5) follows from the result obtained by Motornyi in [15], which establishes exact asymptotic equalities between the upper bounds in the uniform and integral metrics for the deviations of functions of the class H_p^1, $p = 1, \infty$, from the operators with positive kernels generated by linear methods of summation of Fourier series. The theorem is proved.

Corollary. *Since* $\mathcal{E}\left(H_p^1, A_3(r,\theta)\right)_p = \mathcal{E}\left(H_p^2, A_3(r,\theta)\right)_p$, $p = 1, \infty$ *(see, e.g., [11]), the quantity* $\mathcal{E}\left(H_p^2, A_3(r,\theta)\right)_p$ *can be expanded in the asymptotic series presented on the right-hand side of equality (5).*

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